Application Of Derivatives

1. The derivative of sin (x^2) w.r.t. x, at x = $\sqrt{\pi}$ is : (2024)

(A) 1

(B) – 1

(C) $-2\sqrt{\pi}$

(D) $2\sqrt{\pi}$

Ans. (C) $-2\sqrt{\pi}$

2. Show that the function f(x) = $4x^3 - 18x^2 + 27x - 7$ has neither maxima nor minima. (2024)

Ans. $f'(x) = 12x^2 - 36x + 27$

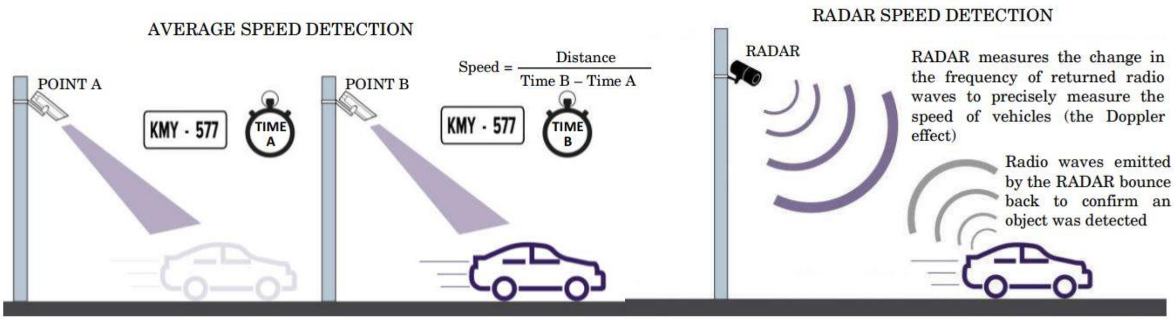
$= 3(2x - 3)^2 \geq 0$ for all $x \in R$

∴ f is increasing on R .

Hence f(x) does not have maxima or minima.

3. Case Study Based Question : (2024)

The traffic police has installed Over Speed Violation Detection (OSVD) system at various locations in a city. These cameras can capture a speeding vehicle from a distance of 300 m and even function in the dark.



A camera is installed on a pole at the height of 5 m. It detects a car travelling away from the pole at the speed of 20 m/s. At any point, x m away from the base of the pole, the angle of elevation of the speed camera from the car C is θ.

On the basis of the above information, answer the following questions :

(i) Express θ in terms of height of the camera installed on the pole and x.

(ii) Find dθ/dx.

(iii) (a) Find the rate of change of angle of elevation with respect to time at an instant when the car is 50 m away from the pole.

(b) If the rate of change of angle of elevation with respect to time of another car at a distance of 50 m from the base of the pole is 3/101 rad/s, then find the speed of the car.

Ans.

(i) $\tan\theta = \frac{5}{x} \Rightarrow \theta = \tan^{-1}\left(\frac{5}{x}\right)$

(ii) $\frac{d\theta}{dx} = \frac{-5}{5^2 + x^2}$

(iii) (a) $\frac{d\theta}{dt} = \frac{d\theta}{dx} \times \frac{dx}{dt} = \frac{-5}{5^2 + x^2} \times 20\Big]_{x=50}$

$= \frac{-100}{2525}$ or $\frac{-4}{101}$ rad/s

(b) $\frac{d\theta}{dt} = \frac{d\theta}{dx} \times \frac{dx}{dt} \Rightarrow \frac{3}{101} = \frac{-5}{5^2 + x^2}\Big]_{x=50} \times \frac{dx}{dt}$

$\Rightarrow \frac{3}{101} = \frac{-5}{2525} \times \frac{dx}{dt} \Rightarrow \frac{dx}{dt} = -15\ m/s$

Hence the speed is 15 m/s

Previous Years' CBSE Board Questions

6.2 Rate of Change of Quantities

VSA (1 mark)

1. The radius of a circle is increasing at the uniform rate of 3 cm/sec. At the instant when the radius of the circle is 2 cm, its area increases at the rate of ________ cm^2/s. *(2020)* Ap

2. The rate of change of the area of a circle with respect to its radius r, when r = 3 cm, is ________. *(2020)*

SA I (2 marks)

3. The total cost $C(x)$ associated with the production of x units of an item is given by $C(x) = 0.005x^3 - 0.02x^2 + 30x + 5000$. Find the marginal cost when 3 units are produced, where by marginal cost we mean the instantaneous rate of change of total cost at any level of output. *(2018)* Ap

4. The volume of a sphere is increasing at the rate of 3 cubic centimeter per second. Find the rate of increase of its surface area, when the radius is 2 cm. *(Delhi 2017)*

5. The volume of a cube is increasing at the rate of 9 cm^3/s. How fast is its surface area increasing when the length of an edge is 10 cm? *(NCERT, AI 2017)* Ev

LA I (4 marks)

6. A ladder 13 m long is leaning against a vertical wall. The bottom of the ladder is dragged away from the wall along the ground at the rate of 2 cm/sec. How fast is the height on the wall decreasing when the foot of the ladder is 5 m away from the wall? *(AI 2019)*

7. The side of an equilateral triangle is increasing at the rate of 2 cm/s. At what rate is its area increasing when the side of the triangle is 20 cm? *(Delhi 2015)* Ev

8. The sides of an equilateral triangle are increasing at the rate of 2 cm/sec. Find the rate at which the area increases, when the side is 10 cm. *(AI 2014C)*

6.3 Increasing and Decreasing Functions

MCQ

9. The interval in which the function $f(x) = 2x^3 + 9x^2 + 12x - 1$ is decreasing, is
   (a) $(-1, \infty)$ (b) $(-2, -1)$ (c) $(-\infty, -2)$ (d) $[-1, 1]$ *(2023)*

10. The function $f(x) = x^3 + 3x$ is increasing in interval
    (a) $(-\infty, 0)$ (b) $(0, \infty)$ (c) R (d) $(0, 1)$ *(2023)*

11. The interval, in which function $y = x^3 + 6x^2 + 6$ is increasing, is
    (a) $(-\infty,-4)\cup(0,\infty)$ (b) $(-\infty,-4)$
    (c) $(-4, 0)$ (d) $(-\infty, 0)\cup(4,\infty)$ *(Term I, 2021-22)*

12. The function $(x - \sin x)$ decreases for
    (a) all x (b) $x < \frac{\pi}{2}$
    (c) $0 < x < \frac{\pi}{4}$ (d) no value of x *(Term I, 2021-22)*

VSA (1 mark)

13. Find the interval in which the function f given by $f(x) = 7 - 4x - x^2$ is strictly increasing. *(2020)*

SA I (2 marks)

14. Find the interval in which the function $f(x) = 2x^3 - 3x$ is strictly increasing. *(2023)*

15. Show that the function $f(x) = 4x^3 - 18x^2 + 27x - 7$ is always increasing on R. *(Delhi 2017)*

16. Show that the function $f(x) = x^3 - 3x^2 + 6x - 100$ is increasing on R. *(AI 2017)* Ap

LA I (4 marks)

17. Find whether the function $f(x) = \cos\left(2x+\frac{\pi}{4}\right)$; is increasing or decreasing in the interval $\frac{3\pi}{8} < x < \frac{5\pi}{8}$. *(2019)* U

18. Find the intervals in which the function $f(x) = \frac{x^4}{4} - x^3 - 5x^2 + 24x + 12$ is
    (a) strictly increasing
    (b) strictly decreasing *(2018)* Ap

19. Find the intervals in which the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is
    (a) strictly increasing
    (b) strictly decreasing *(Delhi 2014)*

20. Find the value(s) of x for which $y = [x(x-2)]^2$ is an increasing function. *(AI 2014)*

21. Find the intervals in which the function $f(x) = \frac{3}{2}x^4 - 4x^3 - 45x^2 + 51$ is
    (i) strictly increasing
    (ii) strictly decreasing *(Foreign 2014)* Ap

22. Find the intervals in which the function $f(x) = \frac{3}{10}x^4 - \frac{4}{5}x^3 - 3x^2 + \frac{36}{5}x + 11$ is
    (a) strictly increasing
    (b) strictly decreasing. *(NCERT, AI 2014C)* Ev

LA II (5/6 marks)

23. Find the intervals on which the function $f(x) = (x - 1)^3 (x - 2)^2$ is (a) strictly increasing (b) strictly decreasing. *(2020)*

24. Find the intervals in which the function f defined as $f(x) = \sin x + \cos x, 0 \le x \le 2\pi$ is strictly increasing or decreasing. *(2020)* U

25. Find the intervals in which $f(x) = \sin 3x - \cos 3x, 0 < x < \pi$, is strictly increasing or strictly decreasing. *(Delhi 2016)* An

26. Prove that the function f defined by $f(x) = x^2 - x + 1$ is neither increasing nor decreasing in $(-1, 1)$. Hence, find the intervals in which $f(x)$ is (i) strictly increasing (ii) strictly decreasing. *(Delhi 2014C)*

6.4 Maxima and Minima

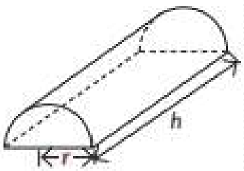
MCQ

27. The value of x for which $(x - x^2)$ is maximum, is
(a) 3/4 (b) 1/2 (c) 1/3 (d) 1/4
(Term I, 2021-22) U

28. A wire of length 20 cm is bent in the form of a sector of a circle. The maximum area that can be enclosed by the wire is
(a) 20 sq.cm (b) 25 sq.cm
(c) 10 sq.cm (d) 30 sq.cm
(Term I, 2021-22)

Case study-Some young entrepreneur started a industry "young achievers" for casting metal into various shapes. They put up an advertisement online stating the same and expecting order to cast metal for toys, sculptures, decorative pieces and more.

A group of friends wanted to make innovative toys and hence contacted the "young achievers" to order them to cast metal into solid half cylinders with a rectangular base and semi-circular ends.



Based on the above information, answer the following questions (29 to 33) :

29. The volume (V) of the casted half cylinder will be
(a) $\pi r^2 h$ (b) $\frac{1}{3}\pi r^2 h$
(c) $\frac{1}{2}\pi r^2 h$ (d) $\pi r^2(r + h)$
(Term I, 2021-22)

30. The total surface area (S) of the casted half cylinder will be
(a) $\pi rh + 2\pi r^2 + rh$ (b) $\pi rh + \pi r^2 + 2rh$
(c) $2\pi rh + \pi r^2 + 2rh$ (d) $\pi rh + \pi r^2 + rh$
(Term I, 2021-22) Ev

31. The total surface area S can be expressed in terms of V and r as
(a) $2\pi r + \frac{2V(\pi+2)}{\pi r}$ (b) $\pi r + \frac{2V}{\pi r}$
(c) $\pi r^2 + \frac{2V(\pi+2)}{\pi r}$ (d) $2\pi r^2 + \frac{2V(\pi+2)}{\pi r}$
(Term I, 2021-22) Ap

32. For the given half-cylinder of volume V, the total surface area S is minimum, when
(a) $(\pi + 2) V = \pi^2 r^3$ (b) $(\pi + 2) V = \pi^2 r^2$
(c) $2(\pi + 2) V = \pi^2 r^3$ (d) $(\pi + 2) V = \pi^2 r$
(Term I, 2021-22)

33. The ratio $h : 2r$ for which S to be minimum will be equal to
(a) $2\pi : \pi + 2$ (b) $2\pi : \pi + 1$
(c) $\pi : \pi + 1$ (d) $\pi : \pi + 2$
(Term I, 2021-22) Cr

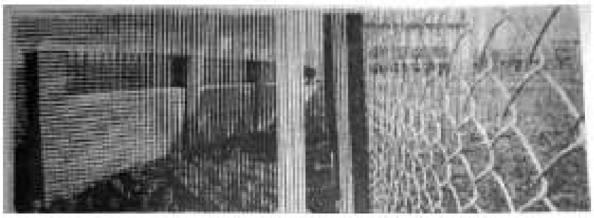
VSA (1 mark)

34. The absolute minimum value of $f(x) = 2 \sin x$ in $\left[0, \frac{3\pi}{2}\right]$ is ______. *(2020)*

35. The least value of the function $f(x) = ax + \frac{b}{x}$ $(a > 0, b > 0, x > 0)$ is ______. *(2020)*

LA I (4 marks)

36. **Case-study :** Sooraj's father wants to construct a rectangular garden using a brick wall on one side of the garden and wire fencing for the other three sides as shown in the figure. He has 200 metres of fencing wire.



Based on the above information, answer the following questions :
(i) Let 'x' metres denote the length of the side of the garden perpendicular to the brick wall and 'y' metres denote the length of the side parallel to the brick wall. Determine the relation representing the total length of fencing wire and also write $A(x)$, the area of the garden.
(ii) Determine the maximum value of $A(x)$. *(2023)*

37. An open tank with a square base and vertical sides is to be constructed from a metal sheet so as to hold a given quantity of water. Show that the cost of material will be least when depth of the tank is half of its width. If the cost is to be borne by nearby settled lower income families, for whom water will be provided, what kind of value is hidden in this question? *(2018)* Ap

LA II (5 / 6 marks)

38. The median of an equilateral triangle is increasing at the rate of $2\sqrt{3}$ cm/s. Find the rate at which its side is increasing. *(2023)*

39. Sum of two numbers is 5. If the sum of the cubes of these numbers is least, then find the sum of the squares of these numbers. *(2023)*

40. Show that the height of the right circular cylinder of greatest volume which can be inscribed in a right

circular cone of height h and radius r is one-third of the height of the cone, and the greatest volume of the cylinder is $\frac{4}{9}$ times the volume of the cone. (2020)

41. Find the minimum value of $(ax + by)$, where $xy = c^2$. (2020, Foreign 2015)

42. Amongst all open (from the top) right circular cylindrical boxes of volume 125π cm^3, find the dimensions of the box which has the least surface area. (2020)

43. Find the dimensions of the rectangle of perimeter 36 cm which will sweep out a volume as large as possible, when revolved about one of its side. Also, find the maximum volume. (2020) Ap

44. Prove that the radius of the right circular cylinder of greatest curved surface area which can be inscribed in a given cone is half of that of the cone. (2020C)

45. Show that the height of the cylinder of maximum volume that can be inscribed in a sphere of radius R is $\frac{2R}{\sqrt{3}}$. Also find the maximum volume. (2019)

46. Show that the height of a cylinder, which is open at the top, having a given surface area and greatest volume, is equal to the radius of its base. (2019)

47. A tank with rectangular base and rectangular sides, open at the top is to be constructed so that its depth is 2 m and volume is 8 m^3. If building of tank costs ₹ 70 per square metre for the base and ₹ 45 per square metre for the sides, what is the cost of least expensive tank? (NCERT, Delhi 2019) Cr

48. Find the area of the greatest rectangle that can be inscribed in an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. (AI 2019)

49. Show that the right circular cylinder of given surface area and maximum volume is such that its height is equal to the diameter of the base. (2019C)

50. If the sum of lengths of the hypotenuse and a side of a right angled triangle is given, show that the area of the triangle is maximum, when the angle between them is $\frac{\pi}{3}$. (NCERT Exemplar, Delhi 2017, AI 2016, 2014)

51. Show that the surface area of a closed cuboid with square base and given volume is minimum, when it is a cube. (AI 2017) Ev

52. Show that the altitude of the right circular cone of maximum volume that can be inscribed in a sphere of radius r is $\frac{4r}{3}$. Also find maximum volume in terms of volume of the sphere. (Delhi 2016, AI 2014) Ap

53. Prove that the least perimeter of an isosceles triangle in which a circle of radius r can be inscribed is $6\sqrt{3}r$. (AI 2016)

54. The sum of the surface areas of a cuboid with sides x, $2x$ and $\frac{x}{3}$ and a sphere is given to be constant. Prove that the sum of their volumes is minimum, if x is equal to three times the radius of sphere. Also find the minimum value of the sum of their volumes. (Foreign 2016) Ap

OR

The sum of surface areas of a sphere and a cuboid with sides $\frac{x}{3}$, x and $2x$, is constant. Show that the sum of their volumes is minimum if x is equal to three times the radius of sphere. (AI 2015C)

55. Find the local maxima and local minima of the function $f(x) = \sin x - \cos x$, $0 < x < 2\pi$. Also find the local maximum and local minimum values. (Delhi 2015)

56. Find the coordinates of a point of the parabola $y = x^2 + 7x + 2$ which is closest to the straight line $y = 3x - 3$. (Foreign 2015)

57. A tank with rectangular base and rectangular sides open at the top is to be constructed so that its depth is 3 m and volume is 75 m^3. If building of tank costs ₹ 100 per square metre for the base and ₹ 50 per square metre for the sides, find the cost of least expensive tank. (Delhi 2015C) Ap

58. A point on the hypotenuse of a right triangle is at distance 'a' and 'b' from the sides of the triangle. Show that the minimum length of the hypotenuse is $(a^{2/3} + b^{2/3})^{3/2}$. (NCERT, Delhi 2015C)

59. Of all the closed right circular cylindrical cans of volume 128π cm^3, find the dimensions of the can which has minimum surface area. (Delhi 2014)

60. Show that the semi vertical angle of the cone of the maximum volume and of given slant height is $\cos^{-1}\frac{1}{\sqrt{3}}$. (Delhi 2014) Ev

61. Prove that the semi vertical angle of the right circular cone of given volume and least curved surface area is $\cot^{-1}\sqrt{2}$. (Delhi 2014)

62. The sum of the perimeters of a circle and a square is k, where k is some constant. Prove that the sum of their areas is least when the side of the square is equal to the diameter of the circle. (Foreign 2014, Delhi 2014C)

63. Show that a cylinder of a given volume which is open at the top has minimum total surface area, when its height is equal to the radius of its base. (Foreign 2014) Ap

64. A window is of the form of a semi-circle with a rectangle on its diameter. The total perimeter of the window is 10 m. Find the dimension of the window to admit maximum light through the whole opening. (Foreign 2014)

65. AB is a diameter of a circle and C is any point on the circle. Show that the area of ΔABC is maximum, when it is isosceles. (AI 2014C)

66. Find the point P on the curve $y^2 = 4ax$ which is nearest to the point $(11a, 0)$. (AI 2014C) Ev

67. If the length of three sides of a trapezium other than base is 10 cm each, then find the area of the trapezium when it is maximum. (NCERT, AI 2014C) Ap

6.2 Rate of Change of Quantities

SA I (2 marks)

1. A man 1.6 m tall walks at the rate of 0.3 m/sec away from a street light that is 4 m above the ground. At what rate is the tip of his shadow moving? At what rate is his shadow lengthening? *(2022-23)*

6.3 Increasing and Decreasing Functions

MCQ

2. Find the intervals in which the function f given by $f(x) = x^2 - 4x + 6$ is strictly increasing.
   (a) $(-\infty, 2) \cup (2, \infty)$ (b) $(2, \infty)$
   (c) $(-\infty, 2)$ (d) $(-\infty, 2] \cup (2, \infty)$
   (Term I, 2021-22) Ev

3. The real function $f(x) = 2x^3 - 3x^2 - 36x + 7$ is
   (a) Strictly increasing in $(-\infty, -2)$ and strictly decreasing in $(-2, \infty)$
   (b) Strictly decreasing in $(-2, 3)$
   (c) Strictly decreasing in $(-\infty, 3)$ and strictly increasing in $(3, \infty)$
   (d) Strictly decreasing in $(-\infty, -2) \cup (3, \infty)$
   (Term I, 2021-22)

4. The value of b for which the function $f(x) = x + \cos x + b$ is strictly decreasing over R is
   (a) $b < 1$ (b) No value of b exists
   (c) $b \le 1$ (d) $b \ge 1$
   (Term I, 2021-22) Ap

SA II (3 marks)

5. Find the intervals in which the function f given by $f(x) = \tan x - 4x$, $x \in \left(0, \frac{\pi}{2}\right)$ is
   (a) strictly increasing (b) strictly decreasing
   (2020-21)

6.4 Maxima and Minima

MCQ

6. The least value of the function $f(x) = 2\cos x + x$ in the closed interval $\left[0, \frac{\pi}{2}\right]$ is
   (a) 2 (b) $\frac{\pi}{6} + \sqrt{3}$
   (c) $\frac{\pi}{2}$
   (d) The least value does not exist.
   (Term I, 2021-22)

7. The area of a trapezium is defined by function f and given by $f(x) = (10 + x)\sqrt{100 - x^2}$, then the area when it is maximised is
   (a) 75 cm^2 (b) $7\sqrt{3}\text{ cm}^2$
   (c) $75\sqrt{3}\text{ cm}^2$ (d) 5 cm^2
   (Term I, 2021-22) Cr

8. The maximum value of $[x(x - 1) + 1]^{1/3}$, $0 \le x \le 1$ is
   (a) 0 (b) $\frac{1}{2}$ (c) 1 (d) $\sqrt[3]{\frac{1}{3}}$
   (Term I, 2021-22)

Case Study : The fuel cost per hour for running a train is proportional to the square of the speed it generates in km per hour. If the fuel costs ₹ 48 per hour at speed 16 km per hour and the fixed charges to run the train amount is ₹ 1200 per hour.

Assume the speed of the train as v km/h.
Based on the given information, answer the following questions (9 to 13).

9. Given that the fuel cost per hour is k times the square of the speed the train generates in km/h, the value of k is
   (a) $\frac{16}{3}$ (b) $\frac{1}{3}$ (c) 3 (d) $\frac{3}{16}$

10. If the train has travelled a distance of 500 km, then the total cost of running the train is given by function
    (a) $\frac{15}{16}v + \frac{600000}{v}$ (b) $\frac{375}{4}v + \frac{600000}{v}$
    (c) $\frac{5}{16}v^2 + \frac{150000}{v}$ (d) $\frac{3}{16}v + \frac{6000}{v}$

11. The most economical speed to run the train is
    (a) 18 km/h (b) 5 km/h
    (c) 80 km/h (d) 40 km/h

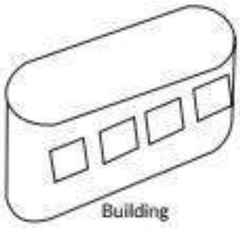
12. The fuel cost for the train to travel 500 km at the most economical speed is

(a) ₹ 3750 (b) ₹ 750
(c) ₹ 7500 (d) ₹ 75000

13. The total cost of the train to travel 500 km at the most economical speed is
(a) ₹ 3750 (b) ₹ 75000
(c) ₹ 7500 (d) ₹ 15000
(Term I, 2021-22)

Case study based questions are compulsory. Attempt any 4 sub parts from question. Each sub-part carrie 1 mark.

14. An architect designs a building for a multi-national company. The floor consists of a rectangular region with semicircular ends having a perimeter of 200 m as shown below:



Based on the above information answer the following:

(i) If x and y represents the length and breadth of the rectangular region, then the relation between the variables is
(a) $x + \pi y = 100$ (b) $2x + \pi y = 200$
(c) $\pi x + y = 50$ (d) $x + y = 100$

(ii) The area of the rectangular region A expressed as a function of x is
(a) $\frac{2}{\pi}(100x - x^2)$ (b) $\frac{1}{\pi}(100x - x^2)$
(c) $\frac{x}{\pi}(100 - x)$ (d) $\pi y^2 + \frac{2}{\pi}(100x - x^2)$

(iii) The maximum value of area A is
(a) $\frac{\pi}{3200}\text{m}^2$ (b) $\frac{3200}{\pi}\text{m}^2$
(c) $\frac{5000}{\pi}\text{m}^2$ (d) $\frac{1000}{\pi}\text{m}^2$

(iv) The CEO of the multi-national company is interested in maximizing the area of the whole floor including the semi-circular ends. For this to happen the value of x should be
(a) 0 m (b) 30 m
(c) 50 m (d) 80 m

(v) The extra area generated if the area of the whole floor is maximized is
(a) $\frac{3000}{\pi}\text{m}^2$ (b) $\frac{5000}{\pi}\text{m}^2$
(c) $\frac{7000}{\pi}\text{m}^2$
(d) No change, Both areas are equal *(2020-21)* Cr

LA I **(4 marks)**

15. **Case-Study :** Read the following passage and answer the questions given below.

In an elliptical sport field the authority wants to design a rectangular soccer field with the maximum possible area. The sport field is given by the graph of $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

(i) If the length and the breadth of the rectangular field be $2x$ and $2y$ respectively, then find the area function in terms of x.

(ii) Find the critical point of the function.

(iii) Use First Derivative Test to find the length $2x$ and width $2y$ of the soccer field (in terms of a and b) that maximize its area.

OR

Use Second Derivative Test to find the length $2x$ and width $2y$ of the soccer field (in terms of a and b) that maximize its area. *(2022-23)*

Detailed SOLUTIONS

Previous Years' CBSE Board Questions

1. Let r be the radius and A be the area of circle.

Given that $\frac{dr}{dt} = 3\text{ cm/sec}$ ...(i)

We know that, area of circle $A = \pi r^2$

$\therefore \frac{dA}{dt} = 2\pi r \frac{dr}{dt} = 2\pi r \cdot 3$ [Using (i)]
$= 6\pi r$

$\therefore \left(\frac{dA}{dt}\right)_{r=2\text{ cm}} = 12\pi\text{ cm}^2/\text{s}$

2. Let r be the radius and A be the area of circle.

We know that, area of circle, $A = \pi r^2$

$\therefore \frac{dA}{dr} = 2\pi r \Rightarrow \left(\frac{dA}{dr}\right)_{r=3} = 6\pi\text{ cm}$

3. We have, $C(x) = 0.005x^3 - 0.02x^2 + 30x + 5000$

$\Rightarrow \frac{dC}{dx} = 0.015x^2 - 0.04x + 30$

Now, $\left(\frac{dC}{dx}\right)_{x=3} = 0.015\times3^2 - 0.04\times3 + 30 = 30.015$

4. Let r, S and V respectively be the radius, surface area and volume of sphere at any time t.

Given, $\frac{dV}{dt} = 3\,\text{cm}^3/\text{sec}$

We know that, volume of sphere $V = \frac{4}{3}\pi r^3$

$\Rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

$\Rightarrow \frac{dr}{dt} = \frac{3}{4\pi r^2}$ cm/sec

We know that, surface area of sphere $S = 4\pi r^2$

$\Rightarrow \frac{dS}{dt} = 8\pi r\frac{dr}{dt} = 8\pi r\left(\frac{3}{4\pi r^2}\right) \Rightarrow \frac{dS}{dt} = \frac{6}{r}$

$\therefore \left(\frac{dS}{dt}\right)_{r=2\text{cm}} = \frac{6}{2} = 3\,\text{cm}^2/\text{sec}$

5. Let l be the length of an edge and V be the volume of a cube respectively.

Given, $\frac{dV}{dt} = 9\text{cm}^3/\text{s}$ and $l = 10\text{cm}$

We know that, volume of cube $(V) = l^3$

$\therefore \frac{dV}{dt} = \frac{d}{dt}(l)^3 \Rightarrow 9 = 3l^2\frac{dl}{dt}$

$\Rightarrow \frac{dl}{dt} = \frac{3}{l^2}$ ...(i)

And, surface area of cube $(A) = 6l^2$

$\therefore \frac{dA}{dt} = \frac{d}{dt}(6l^2) = 12l\frac{dl}{dt} = 12l\times\frac{3}{l^2}$ (From (i))

$= \frac{36}{l}$

$\therefore \left(\frac{dA}{dt}\right)_{l=10} = \frac{36}{10} = 3.6\text{cm}^2/\text{s}$

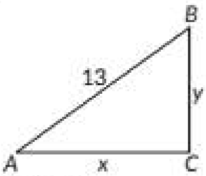
6. Let foot of the ladder is at a distance x m from the wall and height on the wall is y m.

Here, $x^2 + y^2 = (13)^2$ [Using Pythagoras theorem]

Differentiating with respect to t, we get

$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$

$\Rightarrow \frac{dy}{dt} = \frac{-x}{y}\frac{dx}{dt}$



When $x = 5$ m, $y^2 = (13)^2 - (5)^2 = 169 - 25 = 144$

$\therefore y = 12$ m

Also, $\frac{dx}{dt} = 2$ cm/sec [Given]

$\therefore \frac{dy}{dt} = \frac{-5}{12}\times2 = \frac{-5}{6}$ cm/sec

7. Let 'a' be the side of an equilateral triangle.

Then $\frac{da}{dt} = 2\text{cm/sec}$

Let 'A' be the area of an equilateral triangle, then

$A = \frac{\sqrt3}{4}a^2 \Rightarrow \frac{dA}{dt} = 2\times\frac{\sqrt3}{4}a\frac{da}{dt} = \frac{\sqrt3}{2}a\frac{da}{dt}$

$\therefore \left(\frac{dA}{dt}\right)_{a=20} = \frac{\sqrt3}{2}\times20\times2 = 20\sqrt3\,\text{cm}^2/\text{sec}$

8. Let 'a' be the side of an equilateral triangle.

Then $\frac{da}{dt} = 2\text{cm/sec}$

Let 'A' be the area of an equilateral triangle, then

$A = \frac{\sqrt3}{4}a^2 \Rightarrow \frac{dA}{dt} = 2\times\frac{\sqrt3}{4}a\frac{da}{dt} = \frac{\sqrt3}{2}a\frac{da}{dt}$

$\therefore \left(\frac{dA}{dt}\right)_{a=10} = \frac{\sqrt3}{2}\times10\times2 = 10\sqrt3\,\text{cm}^2/\text{sec}$

Concept Applied

- Power rule of derivative : $\frac{d}{dx}(x^n) = nx^{n-1}$

9. (b): We have, $f(x) = 2x^3 + 9x^2 + 12x - 1$

$\Rightarrow f'(x) = 6x^2 + 18x + 12$

For decreasing, $f'(x) < 0$

$\therefore 6x^2 + 18x + 12 < 0$

$\Rightarrow x^2 + 3x + 2 < 0 \Rightarrow (x+1)(x+2) < 0 \Rightarrow -2 < x < -1$

So, $f(x)$ is decreasing, if $x \in (-2, -1)$.

10. (c): $f(x) = x^3 + 3x$

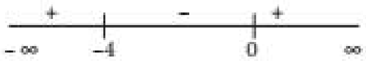
For increasing, we must have $f'(x) > 0$

$\therefore f'(x) = 3x^2 + 3 > 0 \Rightarrow 3(x^2+1) > 0$

$\Rightarrow x^2 + 1 > 0$, which is true $\forall x \in R$.

11. (a): Given, $y = x^3 + 6x^2 + 6 \Rightarrow \frac{dy}{dx} = 3x^2 + 12x$

For increasing, $\frac{dy}{dx} > 0 \Rightarrow 3x^2 + 12x > 0 \Rightarrow 3x(x+4) > 0$



So, y is strictly increasing in $(-\infty, -4) \cup (0, \infty)$.

12. (d): Let $f(x) = x - \sin x$

Differentiating w.r.t. x, we get $f'(x) = 1 - \cos x$

For function to be decreasing, $f'(x) < 0$

$\Rightarrow 1 - \cos x < 0 \Rightarrow \cos x > 1$,

which is not possible, because maximum value of cosx is 1.

$\therefore f(x) = (x - \sin x)$ doesn't decrease at any value of x.

13. Let $y = f(x) = 7 - 4x - x^2$

$\therefore \frac{dy}{dx} = -4 - 2x$

For strictly increasing, $\frac{dy}{dx} > 0$

$\Rightarrow -4 - 2x > 0 \Rightarrow x < -2$

$\therefore$ Required interval is $(-\infty, -2)$.

14. Let $y = f(x) = 2x^3 - 3x \quad \therefore \frac{dy}{dx} = 6x^2 - 3$

For strictly increasing, $\frac{dy}{dx} > 0$

$\Rightarrow 6x^2 - 3 > 0 \Rightarrow x^2 > \frac{1}{2}$

So, $f(x)$ is strictly increasing in $x \in \left(-\infty, \frac{-1}{\sqrt2}\right) \cup \left(\frac{1}{\sqrt2}, \infty\right)$.

15. We have, $f(x) = 4x^3 - 18x^2 + 27x - 7$

$\Rightarrow f'(x) = 12x^2 - 36x + 27$

$= 12\left(x^2-3x+\frac{9}{4}-\frac{9}{4}\right)+27$

$=12\left(x-\frac{3}{2}\right)^2-27+27=12\left(x-\frac{3}{2}\right)^2 \geq 0\ \forall x \in R$

Hence, $f(x)$ is always increasing on R.

16. We have, $f(x) = x^3 - 3x^2 + 6x - 100$ ...(i)

Differentiating (i) w.r.t. x, we get

$f'(x) = 3x^2 - 6x + 6$

$= 3(x^2 - 2x + 1) + 3 = 3(x - 1)^2 + 3 > 0$

($\because$ For all values of x, $(x - 1)^2$ is always positive)

$\therefore \quad f'(x) > 0$

So, $f(x)$ is an increasing function on R.

Answer Tips

- If $f'(x) > 0 \Rightarrow f$ is strictly increasing function.

17. We have, $f(x)=\cos\left(2x+\frac{\pi}{4}\right)$

$\therefore \quad f'(x)=-2\sin\left(2x+\frac{\pi}{4}\right)$

Given, $\frac{3\pi}{8}<x<\frac{5\pi}{8}$

$\Rightarrow \frac{3\pi}{4}<2x<\frac{5\pi}{4} \Rightarrow \frac{\pi}{4}+\frac{3\pi}{4}<2x+\frac{\pi}{4}<\frac{5\pi}{4}+\frac{\pi}{4}$

$\Rightarrow \pi<2x+\frac{\pi}{4}<\frac{3\pi}{2} \Rightarrow \sin\left(2x+\frac{\pi}{4}\right)<0$

[$\because$ sin function is negative in IIIrd and IVth quadrant]

$\Rightarrow -2\sin\left(2x+\frac{\pi}{4}\right)>0$

$\Rightarrow \quad f'(x) > 0$

Hence, $f(x)$ is increasing in $\left(\frac{3\pi}{8},\frac{5\pi}{8}\right)$.

18. We have, $f(x) = \frac{x^4}{4} - x^3 - 5x^2 + 24x + 12$ ...(i)

$f(x)$ being polynomial function is continuous and derivable on R.

Differentiating (i) w.r.t. x, we get

$f'(x)=\frac{4x^3}{4}-3x^2-10x+24$

$= x^3 - 3x^2 - 10x + 24$

$= (x - 2)(x^2 - x - 12) = (x - 2)(x - 4)(x + 3)$

(a) For strictly increasing, $f'(x) > 0$

$\Rightarrow (x - 2)(x - 4)(x + 3) > 0$

$\Rightarrow x \in (-3, 2) \cup (4, \infty)$

(b) For strictly decreasing, $f'(x) < 0$

$\Rightarrow (x - 2)(x - 4)(x + 3) < 0$

$\Rightarrow x \in (-\infty, -3) \cup (2, 4)$

19. We have, $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$

$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2)$

$\Rightarrow \quad f'(x) = 12x(x + 1)(x - 2)$

Now, $f'(x) = 0$

$\Rightarrow \quad 12x(x + 1)(x - 2) = 0$

$\Rightarrow \quad x = -1, x = 0$ or $x = 2$

Hence, these points divide the whole real line into four disjoint open intervals namely $(-\infty, -1)$, $(-1, 0)$, $(0, 2)$ and $(2, \infty)$.

Interval	Sign of $f'(x)$	Nature of function
$(-\infty, -1)$	$(-)(-)(-) < 0$	Strictly decreasing
$(-1, 0)$	$(-)(+)(-) > 0$	Strictly increasing
$(0, 2)$	$(+)(+)(-) < 0$	Strictly decreasing
$(2, \infty)$	$(+)(+)(+) > 0$	Strictly increasing

(a) $f(x)$ is strictly increasing in $(-1, 0) \cup (2, \infty)$.

(b) $f(x)$ is strictly decreasing in $(-\infty, -1) \cup (0, 2)$.

20. Here, $y = [x(x - 2)]^2 = x^2(x - 2)^2$

$\Rightarrow \quad \frac{dy}{dx}=2x(x-2)^2+2x^2(x-2)$

$=2x(x - 2)(x - 2 + x) = 4x(x - 1)(x - 2)$

For y to be an increasing function, $\frac{dy}{dx} \geq 0$

$\Rightarrow \quad x(x - 1)(x - 2) \geq 0$

Case 1: When $-\infty < x \leq 0$

$\frac{dy}{dx} \leq 0 \Rightarrow y$ is a decreasing function.

Case 2: When $0 \leq x \leq 1$

$\frac{dy}{dx} \geq 0 \Rightarrow y$ is an increasing function.

Case 3: When $1 \leq x \leq 2$

$\frac{dy}{dx} \leq 0 \Rightarrow y$ is a decreasing function.

Case 4: When $2 \leq x \leq \infty$

$\frac{dy}{dx} \geq 0 \Rightarrow y$ is an increasing function.

$\therefore \quad y$ is an increasing function in $[0, 1] \cup [2, \infty)$

21. We have, $f(x)=\frac{3}{2}x^4-4x^3-45x^2+51$ ...(i)

$f(x)$ being a polynomial function is continuous and derivable on R.

Differentiating (i) w.r.t. x, we get

$f'(x)=\frac{3}{2}\times 4x^3-12x^2-90x$

$\Rightarrow \quad f'(x) = 6x^3 - 12x^2 - 90x = 6x(x^2 - 2x - 15)$

$= 6x(x - 5)(x + 3)$

(i) For strictly increasing, $f'(x) > 0$

$\Rightarrow \quad 6x(x - 5)(x + 3) > 0$

$\Rightarrow \quad x \in (-3, 0) \cup (5, \infty)$

(ii) For strictly decreasing, $f'(x) < 0$

$\Rightarrow \quad 6x(x - 5)(x + 3) < 0$

$\Rightarrow \quad x \in (-\infty, -3) \cup (0, 5)$

Answer Tips

- If $f'(x) < 0 \Rightarrow f$ is strictly decreasing function.
- If $f'(x) > 0 \Rightarrow f$ is strictly increasing function.

22. Here, $f(x)=\frac{3}{10}x^4-\frac{4}{5}x^3-3x^2+\frac{36}{5}x+11$

Differentiating w.r.t. x, we get

$f'(x)=\frac{3}{10}\cdot 4x^3-\frac{4}{5}\cdot 3x^2-3\cdot 2x+\frac{36}{5}\cdot 1$

$=\frac{6}{5}(x^3-2x^2-5x+6)\ =\frac{6}{5}(x-1)(x^2-x-6)$

$=\frac{6}{5}(x-1)(x+2)(x-3)$

$\therefore\quad f'(x)=0 \Rightarrow x=-2, 1, 3.$

Hence, the points divide the real line into four disjoint intervals $(-\infty, -2)$, $(-2, 1)$, $(1, 3)$ and $(3, \infty)$.

Interval	Sign of $f'(x)$	Nature of function
$(-\infty, -2)$	$(-)(-)(-)<0$	Strictly decreasing
$(-2, 1)$	$(-)(+)(-)>0$	Strictly increasing
$(1, 3)$	$(+)(+)(-)<0$	Strictly decreasing
$(3, \infty)$	$(+)(+)(+)>0$	Strictly increasing

(a) $f(x)$ is strictly increasing in $(-2, 1)\cup(3, \infty)$.

(b) $f(x)$ is strictly decreasing in $(-\infty, -2)\cup(1, 3)$.

23. We have, $f(x)=(x-1)^3(x-2)^2$ ...(1)

Differentiating equation (1) w.r.t. x, we get

$f'(x)=(x-1)^3\frac{d}{dx}(x-2)^2+(x-2)^2\frac{d}{dx}(x-1)^3$

$=(x-1)^2(x-2)[2(x-1)+3(x-2)]$

$=(x-1)^2(x-2)(2x-2+3x-6)$

$\Rightarrow\quad f'(x)=(x-1)^2(x-2)(5x-8)$

Now put $f'(x)=0$

$\Rightarrow\quad (x-1)^2(x-2)(5x-8)=0 \Rightarrow x=1, 2, 8/5$

The points divide the real line into four disjoint intervals $(-\infty, 1)$, $(1, 8/5)$, $(8/5, 2)$ and $(2, \infty)$.

Interval	Sign of $f'(x)$	Nature of function
$(-\infty, 1)$	$(+)(-)(-)>0$	Strictly increasing
$(1, 8/5)$	$(+)(-)(-)>0$	Strictly increasing
$(8/5, 2)$	$(+)(-)(+)<0$	Strictly decreasing
$(2, \infty)$	$(+)(+)(+)>0$	Strictly increasing

(a) $f(x)$ is strictly increasing in $(-\infty, 1)\cup(1, 8/5)\cup(2, \infty)$.

(b) $f(x)$ is strictly decreasing in $\left(\frac{8}{5}, 2\right)$.

24. The given function is

$f(x)=\sin x+\cos x, 0\le x\le 2\pi$

$\Rightarrow\quad f'(x)=\cos x-\sin x$

Now $f'(x)=0 \Rightarrow \cos x-\sin x=0$

$\Rightarrow \tan x=1$

$\Rightarrow x=\frac{\pi}{4}, \frac{5\pi}{4}$

The points $x=\frac{\pi}{4}$ and $x=\frac{5x}{4}$ divide the interval $[0, 2\pi]$ into three disjoint intervals, $[0, \pi/4)$, $(\pi/4, 5\pi/4)$, $\left(\frac{5\pi}{4}, 2\pi\right]$

Now $f'(x)>0$ in $\left[0, \frac{\pi}{4}\right)$

$\therefore\quad f$ is strictly increasing in $\left[0, \frac{\pi}{4}\right)$.

$f'(x)<0$ in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

$\therefore\quad f$ is strictly decreasing in $\left(\frac{\pi}{4}, \frac{5\pi}{4}\right)$

and $f'(x)>0$ in $\left(\frac{5\pi}{4}, 2\pi\right]$

$\therefore\quad f$ is strictly increasing in $\left(\frac{5\pi}{4}, 2\pi\right]$.

Thus, the function f is strictly increasing in $\left[0, \frac{\pi}{4}\right)\cup\left(\frac{5\pi}{4}, 2\pi\right]$.

25. $f(x)=\sin 3x-\cos 3x$

$\Rightarrow\quad f'(x)=3\cos 3x+3\sin 3x$

$f'(x)=0 \Rightarrow\ 3\cos 3x=-3\sin 3x$

$\Rightarrow\quad \cos 3x=-\sin 3x \Rightarrow \tan 3x=-1$

which gives $3x=\frac{3\pi}{4}$ or $\frac{7\pi}{4}$ or $\frac{11\pi}{4}$

$\Rightarrow\quad x=\frac{\pi}{4}$ or $\frac{7\pi}{12}$ or $\frac{11\pi}{12}$ $[\because\ 0<x<\pi]$

The points $x=\frac{\pi}{4}, x=\frac{7\pi}{12}$ and $x=\frac{11\pi}{12}$ divide the interval $(0, \pi)$ into four disjoint intervals,

$\left(0, \frac{\pi}{4}\right), \left(\frac{\pi}{4}, \frac{7\pi}{12}\right), \left(\frac{7\pi}{12}, \frac{11\pi}{12}\right), \left(\frac{11\pi}{12}, \pi\right)$

Now, $f'(x)>0$ in $\left(0, \frac{\pi}{4}\right)$

$\Rightarrow\quad f$ is strictly increasing in $\left(0, \frac{\pi}{4}\right)$

$f'(x)<0$ in $\left(\frac{\pi}{4}, \frac{7\pi}{12}\right)$

$\Rightarrow\quad f$ is strictly decreasing in $\left(\frac{\pi}{4}, \frac{7\pi}{12}\right)$

$f'(x)>0$ in $\left(\frac{7\pi}{12}, \frac{11\pi}{12}\right)$

$\Rightarrow\quad f$ is strictly increasing in $\left(\frac{7\pi}{12}, \frac{11\pi}{12}\right)$

$f'(x)<0$ in $\left(\frac{11\pi}{12}, \pi\right)$

$\Rightarrow\quad f$ is strictly decreasing in $\left(\frac{11\pi}{12}, \pi\right)$

Hence, f is strictly increasing in the intervals

$\left(0, \frac{\pi}{4}\right)\cup\left(\frac{7\pi}{12}, \frac{11\pi}{12}\right)$

and f is strictly decreasing in the intervals

$\left(\frac{\pi}{4}, \frac{7\pi}{12}\right)\cup\left(\frac{11\pi}{12}, \pi\right)$

26. Here, $f(x)=x^2-x+1; x\in(-1, 1)$

$\Rightarrow\quad f'(x)=2x-1$

$f'(x)=0\Rightarrow x=\frac{1}{2}$

Now $f'(x)=2\left(x-\frac{1}{2}\right)>0$ for $\frac{1}{2}<x<1$

$\Rightarrow\quad f$ is strictly increasing in $\left(\frac{1}{2}, 1\right)$

Also $f'(x)=2\left(x-\frac{1}{2}\right)<0$ for $-1<x<\frac{1}{2}$

$\Rightarrow$ f is strictly decreasing in $\left(-1, \frac{1}{2}\right)$.

Thus f is neither increasing nor decreasing in $(-1, 1)$.

27. (b): Let $f(x) = x - x^2$

$\therefore$ $f'(x) = 1 - 2x$

For critical point, $f'(x) = 0$

$\Rightarrow$ $1 - 2x = 0 \Rightarrow x = 1/2$

Now, at $x = 1/2$, $f''(x) = -2 < 0$

So, $f(x)$ has maximum value at $x = 1/2$.

Concept Applied

- $x = c$ is a point of local maxima if $f'(c) = 0$ and $f''(c) < 0$. The value of $f(c)$ is local maximum value of f.

28. (b): Let r be the radius, θ be the central angle and l be the length of the circular sector.

Given, $l + 2r = 20$

$\Rightarrow$ $r\theta + 2r = 20$ $(\because l = r\theta)$ $\Rightarrow$ $\theta=\frac{20-2r}{r}$

Let A be the area of the circular sector.

$\therefore$ $A = \frac{\pi r^2\theta}{2\pi} = \frac{r^2}{2}\cdot\left(\frac{20-2r}{r}\right)=r(10-r)$

$\Rightarrow$ $\frac{dA}{dr}=10-2r$

For maximum or minimum value of A, we have

$\frac{dA}{dr}=0\Rightarrow r=5$ and $\frac{d^2A}{dr^2}=-2<0$

$\therefore$ Area is maximum at $r = 5$

$\therefore$ Maximum area, $A = 5(10 - 5) = 25\text{ cm}^2$

29. (c): $\because$ Volume of cylinder $= \pi r^2h$

$\therefore$ V = Volume of casted half cylinder $= (1/2)\pi r^2h$

30. (b): Total surface area, $S = \frac{2\pi r(r+h)}{2}+2rh$

$= \pi r^2 + \pi rh + 2rh$

31. (c): Here, $S = \pi r^2+\frac{2V(\pi+2)}{\pi r}$ $\left[\because V=\frac{1}{2}\pi r^2h \Rightarrow \frac{2V}{\pi r}=rh\right]$

32. (a): $\because$ $S = \pi r^2+\frac{2V(\pi+2)}{\pi r}$

$\Rightarrow \frac{dS}{dr}=2\pi r-\frac{2V(\pi+2)}{\pi}\times\frac{1}{r^2}$

For S to be minimum, $\frac{dS}{dr}=0$

$\Rightarrow 2\pi r=\frac{2V(\pi+2)}{\pi r^2}$ $\Rightarrow \pi^2r^3=V(\pi+2)$

33. (d): $\because$ $V=\frac{1}{2}\pi r^2h$ ...(i)

and S will be minimum, when $(\pi + 2)V = \pi^2r^3$

$\Rightarrow V=\frac{\pi^2r^3}{\pi+2}$ ...(ii)

From (i) and (ii), we get

$\Rightarrow \frac{1}{2}\pi r^2h=\frac{\pi^2r^3}{\pi+2}$ $\Rightarrow \pi r^2h(\pi + 2) = 2\pi^2r^3$

$\Rightarrow h(\pi + 2) = 2\pi r$ $\Rightarrow \frac{h}{2r}=\frac{\pi}{\pi+2}$

Thus, required ratio *i.e.*, $h : 2r$ is $\pi : \pi + 2$.

34. Here, $f(x) = 2\sin x$

$\Rightarrow$ $f'(x) = 2\cos x$

Putting $f'(x) = 0$

$\Rightarrow$ $2\cos x = 0 \Rightarrow \cos x = 0$

$\Rightarrow$ $x = \frac{\pi}{2}, \frac{3\pi}{2}$

$\therefore$ $\frac{\pi}{2}, \frac{3\pi}{2}$ are the critical points

At $x = \frac{\pi}{2}$, $f(x) = 2 \times 1 = 2$

At $x = \frac{3\pi}{2}$, $f(x) = 2\sin\left(\pi+\frac{\pi}{2}\right) = -2\sin\frac{\pi}{2} = -2$

Hence, absolute minimum value of $f(x)$ is -2.

35. We have, $f(x) = ax+\frac{b}{x}$

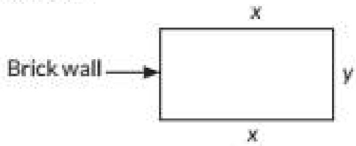
$\therefore$ $f'(x) = a - (b/x^2)$

Putting $f'(x) = 0 \Rightarrow a-\frac{b}{x^2}=0\Rightarrow a=\frac{b}{x^2}\Rightarrow x=\sqrt{\frac{b}{a}}$ (as $x > 0$)

The least value of $f(x)$ is

$f\left(\sqrt{\frac{b}{a}}\right)=a\left(\sqrt{\frac{b}{a}}\right)+\sqrt{ab}=\sqrt{ab}+\sqrt{ab}=2\sqrt{ab}$

36. Given, the length of side of garden perpendicular to the brick wall is x m.



The length of the side parallel to the brick wall is y m.

(i) $2x + y = 200$

We know that area of rectangle is $= l \times b$

$\Rightarrow$ $A(x) = xy = x(200 - 2x) = 200x - 2x^2$

(ii) Since, $A(x) = 200x - 2x^2$ ...(i)

Differentiating (i) w.r.t. x, we get

$\frac{d}{dx}A(x)=200-4x$ ...(ii)

For critical point $\frac{d}{dx}A(x)=0$

$\Rightarrow$ $200 - 4x = 0 \Rightarrow 4x = 200 \Rightarrow x = 50$

Again differentiating (ii) w.r.t. x, we get

$\frac{d^2}{dx^2}A(x)=-4<0$ *i.e.*, area $A(x)$ is maximum at $x = 50$

Hence, maximum area is

$A(50) = 200(50) - 2(50)^2$

$= 10000 - 5000 = 5000\text{ m}^2$

37. Let x be the side of square base and y be the height of the open tank.

$\therefore$ $l = x$, $b = x$ and $h = y$

where l, b and h be the length, breadth and height of tank respectively.

Volume of tank $V = x^2y \Rightarrow y = \frac{V}{x^2}$

The cost of the material will be least if the total surface area is least.

Total surface area of tank $(S) = x^2 + 4xy$

$\Rightarrow S = x^2 + 4x\left(\frac{V}{x^2}\right)$ $\left(\because y = \frac{V}{x^2}\right)$

$\Rightarrow S = x^2 + \frac{4V}{x} \Rightarrow \frac{dS}{dx} = 2x - \frac{4V}{x^2}$

For maxima or minima, $\frac{dS}{dx} = 0$

$\Rightarrow 2x - \frac{4V}{x^2} = 0 \Rightarrow x^3 = 2V \Rightarrow x = 2y$ $(\because V = x^2y)$

Also, $\frac{d^2S}{dx^2} = 2 + \frac{8V}{x^3} > 0$

$\therefore$ Cost of material is least, when $y = \frac{x}{2}$

i.e., the depth of the tank is half of its width.

As the cost is borne by nearby settled lower income families it shows that they are spending money on social welfare so that no body will face the water problem in future. It shows social responsibility.

Concept Applied

- $x = c$ is a point of local minima if $f'(c) = 0$ and $f''(c) > 0$. The value of $f(c)$ is local maximum value of f.

38. Let the side of an equilateral triangle be $2x$ cm, then its median $= \sqrt{3}x$

Let $M = \sqrt{3}x \Rightarrow \frac{dM}{dt} = \sqrt{3}\frac{dx}{dt}$

$\Rightarrow 2\sqrt{3} = \sqrt{3}\frac{dx}{dt} \Rightarrow \frac{dx}{dt} = \frac{2\sqrt{3}}{\sqrt{3}} = 2$ cm/sec

Now, side of triangle $= 2x$

$\Rightarrow s = 2x$

$\Rightarrow \frac{ds}{dt} = 2\frac{dx}{dt} = 2 \times 2$ cm/sec $= 4$ cm/sec

So, side is increasing at the rate of 4 cm/sec.

39. Let the two numbers be x and y.

According to question, we have

$x + y = 5 \Rightarrow y = 5 - x$ ... (i)

Let $p = x^3 + y^3$

$= x^3 + (5 - x)^3$

$= x^3 + 125 - x^3 - 75x + 15x^2$

$\Rightarrow p = 15x^2 - 75x + 125$

Differentiating with respect to x, we get

$\frac{dp}{dx} = 30x - 75$

For minimum, $\frac{dp}{dx} = 0 \Rightarrow 30x - 75 = 0 \Rightarrow x = \frac{5}{2}$

Now, $\frac{d^2p}{dx^2} = 30 > 0$

So, $x^3 + y^3$ is minimum at $x = \frac{5}{2}$

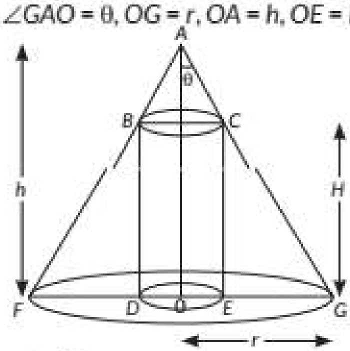
From (i), $y = 5 - \frac{5}{2} = \frac{5}{2}$

So, required value $= x^2 + y^2 = \left(\frac{5}{2}\right)^2 + \left(\frac{5}{2}\right)^2$

$= \frac{25}{4} + \frac{25}{4} = \frac{50}{4} = \frac{25}{2}$

40. Given the right circular cone of fixed height h and semi-vertical angle θ. Let R be the radius of the base and H be the height of the right circular cylinder that can be inscribed in the right circular cone.

In the figure, $\angle GAO = \theta$, $OG = r$, $OA = h$, $OE = R$ and $CE = H$



We have, $r/h = \tan\theta$

$\therefore$ $r = h\tan\theta$ ...(1)

Since, ΔAOG and ΔCEG are similar.

$\therefore$ $\frac{AO}{OG} = \frac{CE}{EG} = \frac{CE}{OG - OE}$

$\therefore$ $\frac{h}{r} = \frac{H}{r - R} \Rightarrow R = r\left(1 - \frac{H}{h}\right)$

Now, volume of cylinder $(V) = \pi R^2 H = \pi r^2\left(1 - \frac{H}{h}\right)^2 H$

For maximum volume, $\frac{dV}{dH} = 0$

$\Rightarrow \pi r^2\left[H \times 2\left(1 - \frac{H}{h}\right) \times -\frac{1}{h} + \left(1 - \frac{H}{h}\right)^2\right] = 0$

$\Rightarrow \pi r^2\left[-\frac{4H}{h} + \frac{3H^2}{h^2} + 1\right] = 0 \Rightarrow \pi r^2\left(\frac{H}{h} - 1\right)\left(\frac{3H}{h} - 1\right) = 0$

$\Rightarrow H = \frac{h}{3}$ $[\because H \neq h]$

So, height of cylinder $= \frac{1}{3}$ of height of cone

Also, maximum volume of cylinder

$= \pi r^2\left(1 - \frac{1}{3}\right)^2 \times \frac{h}{3} = \left(\frac{1}{3}\pi r^2 h\right) \times \frac{4}{9}$

$= \frac{4}{9}$ of volume of cone.

41. Let $u = ax + by$, where $xy = c^2$

$\Rightarrow u = ax + b\left(\frac{c^2}{x}\right)$ ... (i)

Differentiating w.r.t. x, we get

$\frac{du}{dx} = a - \frac{bc^2}{x^2}$ and $\frac{d^2u}{dx^2} = \frac{2bc^2}{x^3}$

For critical points, $\frac{du}{dx} = 0$

$\Rightarrow \frac{ax^2 - bc^2}{x^2} = 0 \Rightarrow x^2 = \frac{bc^2}{a}$

$\therefore \quad x=\pm\sqrt{\frac{b}{a}}c$

At $x=\sqrt{\frac{b}{a}}c,\ \frac{d^2u}{dx^2}=2bc^2\left(\sqrt{\frac{a}{b}}\times\frac{1}{c}\right)^3$

$=\frac{2bc^2}{c^3}\left(\frac{a\sqrt{a}}{b\sqrt{b}}\right)=2\frac{a}{c}\sqrt{\frac{a}{b}}>0$

$\Rightarrow$ u is minimum at $x=c\sqrt{\frac{b}{a}}$

At $x=-\sqrt{\frac{b}{a}}c,\ \frac{d^2u}{dx^2}=-2bc^2\left(\sqrt{\frac{a}{b}}\times\frac{1}{c}\right)<0$

$\Rightarrow$ u is maximum at $x=-\sqrt{\frac{b}{a}}c$

The minimum value of u at $x=\sqrt{\frac{b}{a}}c$ is

$$u=a\left(\sqrt{\frac{b}{a}}c\right)+bc^2\left(\sqrt{\frac{a}{b}}\times\frac{1}{c}\right)=c\sqrt{ab}+\sqrt{bac}=2c\sqrt{ab}$$

42. We have, volume of cylinder = πr^2h

$\pi r^2h = 125\pi$ (given)

$\Rightarrow r^2h = 125$

$\Rightarrow h=\frac{125}{r^2}$ (1)

Surface Area $(S) = 2\pi rh + \pi r^2$

$=2\pi r\cdot\frac{125}{r^2}+\pi r^2$

$\therefore \quad S=\frac{250\pi}{r}+\pi r^2$ (2)

On differentiating (2) w.r.t. to 'r', we get

$\frac{dS}{dr}=\frac{d\left(\frac{250\pi}{r}\right)}{dr}+\frac{d(\pi r^2)}{dr}=-250\pi r^{-2}+2\pi r$

$\frac{dS}{dr}=\frac{-250\pi}{r^2}+2\pi r$ and $\frac{d^2S}{dr^2}=\frac{500\pi}{r^3}+2\pi$

For maximum or minimum value of surface area, we have to put $\frac{dS}{dr}=0$

$\frac{dS}{dr}=0\Rightarrow\frac{-250\pi}{r^2}+2\pi r=0$

$\Rightarrow \quad 2\pi r=\frac{250\pi}{r^2} \Rightarrow r\cdot r^2=\frac{250\pi}{2\pi}$

$\Rightarrow \quad r^3 = 125 \Rightarrow r = 5$ cm

Now, $\left(\frac{d^2S}{dr^2}\right)_{r=5\text{cm}}=\frac{500\pi}{125}+2\pi=6\pi>0$

Hence, the surface area is minimum at $r = 5$ cm

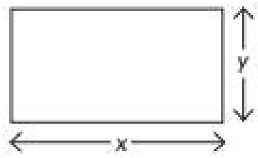
Put $r = 5$ in equation (1), we get

$h=\frac{125}{r^2}=\frac{125}{5\times5}=5\text{cm}$

Hence, the radius and height of the right circular cylindrical box are $r = 5$ cm and $h = 5$ cm respectively.

43. Let length and breadth of rectangle be x and y respectively.

Given, perimeter of rectangle = 36 cm



$\Rightarrow \quad 2x + 2y = 36$

$\Rightarrow \quad x + y = 18$

$\Rightarrow \quad y = 18 - x$ ...(i)

Let rectangle be revolved about its length x.

Then volume of resultant cylinder $(V) = \pi x^2y$

$\Rightarrow \quad V = \pi x^2(18 - x)$ (from (i))

$\Rightarrow \quad V = \pi[18x^2 - x^3]$ ...(ii)

On differentiating (ii) w.r.t. 'x', we get

$\frac{dV}{dx}=\pi(36x-3x^2)$ ...(iii)

Put $\frac{dV}{dx}=0\Rightarrow 3x^2-36x=0$

$\Rightarrow \quad 3x(x - 12) = 0$

$\Rightarrow \quad x = 0, 12 \quad \therefore x = 12$ $(\because x\neq0)$

Again differentiating (iii) w.r.t 'x', we get

$\frac{d^2V}{dx^2}=\pi(36-6x) \Rightarrow \left(\frac{d^2V}{dx^2}\right)_{x=12}=\pi(36-6\times12)=-36\pi<0$

At $x = 12$, volume of resultant cylinder is the maximum therefore the length and breadth of rectangle are 12 cm and 6 cm respectively.

Hence, maximum volume of resultant cylinder,

$(V)_{x=12} = \pi(12^2\times6) = \pi\times144\times6 = 864\ \pi\ \text{cm}^3$

44. Let 'r' and 'h' be the radius and height of right circular cylinder and 'R' and 'H' be the radius and height of cone.

The curved surface area of cylinder, $S = 2\pi rh$

Since, ΔAOC and ΔFEC are similar.

$\therefore \quad \frac{OC}{EC}=\frac{AO}{FE} \Rightarrow \frac{R}{R-r}=\frac{H}{h} \Rightarrow h=\left(\frac{R-r}{R}\right)H$ ...(i)

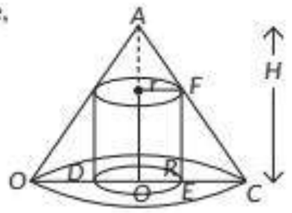
$\therefore \quad S = 2\pi r\left(\frac{R-r}{R}\right)H$ [From (i)]

$\Rightarrow \quad S=\frac{2\pi H(Rr-r^2)}{R}$ ...(ii)

On differentiating (ii) w.r.t. 'r', we have

$\frac{dS}{dr}=\frac{2\pi H}{R}(R-2r)$ ...(iii)

For maximum and minimum value,



put $\frac{dS}{dr}=0,$

$\Rightarrow \quad \frac{2\pi H}{R}(R-2r)=0$

$\Rightarrow \quad R - 2r = 0$

$\Rightarrow r=\frac{R}{2}$

$\therefore \quad \frac{d^2S}{dr^2}=\frac{d}{dr}\left[\frac{2\pi H}{R}(R-2r)\right]$

$=\frac{2\pi H}{R}(0-2)=-\frac{4\pi H}{R}<0$

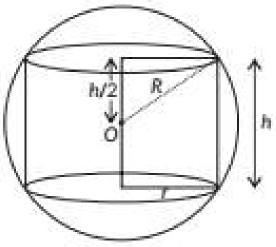
$\therefore \quad \left(\frac{d^2S}{dr^2}\right)_{r=R/2}=\frac{-4\pi H}{R}<0$

Hence for $r=\frac{R}{2}$, surface area is maximum, *i.e.*, surface area is maximum when radius of cylinder is half of that of cone.

45. Let r and h be the base radius and height of cylinder respectively.

$\therefore \quad \left(\frac{h}{2}\right)^2 + r^2 = R^2$ ... (i)

Now, V = Volume of the cylinder inscribed in a sphere $= \pi r^2 h$



$\Rightarrow \quad V = \pi h\left(R^2 - \frac{h^2}{4}\right)$ [Using (i)]

$\Rightarrow \quad V = \pi\left(R^2 h - \frac{h^3}{4}\right)$

Now differentiating w.r.t. h, we get

$\frac{dV}{dh} = \pi\left(R^2 - \frac{3h^2}{4}\right)$ and $\frac{d^2V}{dh^2} = \pi\left(0 - \frac{3}{4}\cdot 2h\right)$

For maximum or minimum,

$\frac{dV}{dh} = 0 \Rightarrow R^2 - \frac{3}{4}h^2 = 0 \Rightarrow h^2 = \frac{4}{3}R^2 \Rightarrow h = \frac{2R}{\sqrt{3}}$

For this value of h, $\frac{d^2V}{dh^2} = -\frac{3}{2}\pi\cdot\frac{2R}{\sqrt{3}} = -\sqrt{3}\pi R < 0$

$\Rightarrow$ V is maximum

Also maximum value of V

$= \pi\cdot\frac{2R}{\sqrt{3}}\left(R^2 - \frac{1}{4}\cdot\frac{4}{3}R^2\right) = \pi\cdot\frac{2R}{\sqrt{3}}\cdot\frac{2}{3}R^2 = \frac{4\pi}{3\sqrt{3}}R^3$ cu.units

Key Points

- Volume of cylinder $= \pi r^2 h$

46. Let r be the radius of base of circular cylinder and h be its height. Let V be the volume and S be total surface area.

$\therefore \quad S = \pi r^2 + 2\pi r h \Rightarrow h = \frac{S - \pi r^2}{2\pi r}$ ...(i)

$V = \pi r^2 h = \pi r^2\left(\frac{S - \pi r^2}{2\pi r}\right)$ [from (i)]

$\therefore \quad V = \frac{1}{2}(Sr - \pi r^3) \Rightarrow \frac{dV}{dr} = \frac{1}{2}(S - 3\pi r^2)$

Put $\frac{dV}{dr} = 0 \Rightarrow \frac{1}{2}(S - 3\pi r^2) = 0 \Rightarrow r = \sqrt{\frac{S}{3\pi}}$

Now, $\frac{d^2V}{dr^2} = \frac{1}{2}(0 - 6\pi r) = -3\pi r$

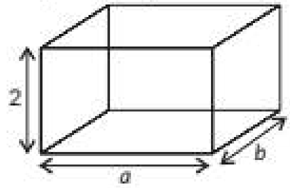
For $r = \sqrt{\frac{S}{3\pi}}, \frac{d^2V}{dr^2} = -3\pi\sqrt{\frac{S}{3\pi}} = -\sqrt{3\pi S} < 0$

$\therefore$ Volume is greatest when $r = \sqrt{\frac{S}{3\pi}}$

From (i), $h = \frac{S - \pi\frac{S}{3\pi}}{2\pi\sqrt{\frac{S}{3\pi}}} = \frac{\frac{2S}{3}}{2\sqrt{\frac{\pi S}{3}}} = \frac{S}{3}\times\frac{\sqrt{3}}{\sqrt{\pi S}} = \sqrt{\frac{S}{3\pi}} = r$

Hence, proved.

47. Let a m and b m be the length and breadth of rectangular tank respectively.



$\therefore$ Volume of tank $= 2ab = 8$ [Given]

$\Rightarrow \quad ab = 4 \Rightarrow b = \frac{4}{a}$ ...(i)

If C is the total cost in rupees, then

$C = 70(ab) + 45(2a + 2b) \times 2$

$\Rightarrow \quad C = 70ab + 90(2a + 2b)$

$\Rightarrow \quad C = 70(a)\left(\frac{4}{a}\right) + 180\left(a + \frac{4}{a}\right)$ [Using (i)]

$\Rightarrow \quad C = 280 + 180a + \frac{720}{a}$ ...(ii)

Differentiating (ii) w.r.t. 'a', we get

$\frac{dC}{da} = 180 - \frac{720}{a^2}$ and $\frac{d^2C}{da^2} = \frac{720\times 2}{a^3}$

For maximum or minimum cost,

$\frac{dC}{da} = 0 \Rightarrow 180 - \frac{720}{a^2} = 0 \Rightarrow a^2 = 4 \Rightarrow a = 2$

For $a = 2, \frac{d^2C}{da^2} > 0 \Rightarrow C$ is least

Using (i), $a = 2$ and $b = 2$

Hence,cost of least expensive tank is

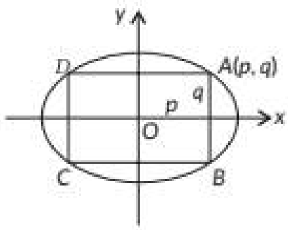
$C = 280 + 360 + 360 = ₹\ 1000$

Key Points

- Remember $\frac{d}{da}\left(\frac{1}{a}\right) = -\frac{1}{a^2}$

48. Let $ABCD$ be a rectangle inscribed in the ellipse,

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



Let $AB = 2q$, $DA = 2p$.

Then coordinates of A are (p, q).

As A lies on the ellipse so

$\frac{p^2}{a^2} + \frac{q^2}{b^2} = 1 \Rightarrow q^2 = b^2\left(1 - \frac{p^2}{a^2}\right)$

Now area A of the rectangle $ABCD = 2p\cdot 2q = 4pq$

$\Rightarrow \quad A^2 = 16p^2q^2$

$= 16p^2\cdot b^2\left(1 - \frac{p^2}{a^2}\right) = 16b^2\left(p^2 - \frac{p^4}{a^2}\right)$

$\therefore \quad \frac{dA^2}{dp}=16b^2\left(2p-\frac{4p^3}{a^2}\right)$ and

$\frac{d^2A^2}{dp^2}=16b^2\left(2-\frac{12p^2}{a^2}\right)$

For A to be max. or min.

$\frac{dA^2}{dp}=0 \Rightarrow 2p-\frac{4p^3}{a^2}=0 \Rightarrow p^2=\frac{a^2}{2}$ $\quad (\because p\neq 0)$

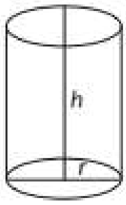
For this value of p^2,

$\frac{d^2A^2}{dp^2}=16b^2(2-12\times\frac{1}{2})=16b^2(-4)<0$

Hence, A^2 is max. $\Rightarrow A$ is max.

$\Rightarrow$ The area of the greatest rectangle inscribed in the ellipse $=4pq=4\cdot\sqrt{\frac{a^2}{2}}\cdot\sqrt{b^2\left(1-\frac{1}{2}\right)}=2ab$ sq.units

49. Let 'r' and 'h' be the radius and height of right circular cylinder.



Surface area of cylinder is given by

$S = 2\pi r^2 + 2\pi rh$

$\Rightarrow \quad 2\pi rh = S - 2\pi r^2$

or $\quad h=\frac{S-2\pi r^2}{2\pi r}$ ...(i)

Volume of cylinder is given by

$V = \pi r^2 h$ ...(ii)

On substituting the value of h from (i) into (ii), we get

$V=\frac{Sr}{2}-\pi r^3$ ...(iii)

On differentiating (iii) with respect to 'r', we get

For maximum and minimum value, put $\frac{dV}{dr}=0$

$\Rightarrow \frac{S}{2}-3\pi r^2=0$

$\Rightarrow \quad S = 6\pi r^2$

or $r^2=\frac{S}{6\pi} \Rightarrow r=\sqrt{\frac{S}{6\pi}}$

Again differentiating (iv) w.r.t. r, we have

$\frac{d^2V}{dr^2}=-6\pi r$

$\left(\frac{d^2V}{dr^2}\right)_{r=\sqrt{\frac{S}{6\pi}}}=-6\pi\left(\sqrt{\frac{S}{6\pi}}\right)<0$

$\therefore$ Volume is maximum when

$r^2=\frac{S}{6\pi}$ or $S=6\pi r^2$

From (i) $h=\frac{6\pi r^2-2\pi r^2}{2\pi r}$

$\Rightarrow \quad h = 2r$

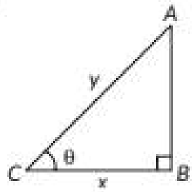
Hence volume is maximum when height is twice the radius, *i.e.*, height is equal to the diameter of base.

50. Let ABC be a right angled triangle with $BC = x$, $AC = y$ such that $x + y = k$, where k is any constant.

Let θ be the angle between the base and the hypotenuse.

Let P be the area of the triangle.

$P=\frac{1}{2}\times BC\times AB=\frac{1}{2}\times x\sqrt{y^2-x^2} \Rightarrow P^2=\frac{x^2}{4}(y^2-x^2)$



$\Rightarrow P^2=\frac{x^2}{4}[(k-x)^2-x^2]$

$\Rightarrow P^2=\frac{k^2x^2-2kx^3}{4}$

Let $Q = P^2$ *i.e.* $Q=\frac{k^2x^2-2kx^3}{4}$

$\therefore \quad P$ is maximum when Q is maximum.

Differentiating Q w.r.t. x, we get

$\frac{dQ}{dx}=\frac{2k^2x-6kx^2}{4}$ ...(i)

For maximum or minimum area,

$\frac{dQ}{dx}=0 \Rightarrow k^2x-3kx^2=0 \Rightarrow x=\frac{k}{3}$

Differentiating (i) w.r.t. x, we get

$\frac{d^2Q}{dx^2}=\frac{2k^2-12kx}{4}$

$\therefore \quad \left[\frac{d^2Q}{dx^2}\right]_{x=\frac{k}{3}}=\frac{-k^2}{2}<0$

Thus, Q is maximum when $x=\frac{k}{3}$

$\Rightarrow \quad P$ is maximum at $x=\frac{k}{3}$

$\therefore \cos\theta=\frac{x}{y}=\frac{k/3}{2k/3}=\frac{1}{2} \Rightarrow \theta=\frac{\pi}{3}$

Answer Tips

- Remember $\cos\theta=\frac{\text{Base}}{\text{Hypotenuse}}$

So, the area of ΔABC is maximum when angle between the hypotenuse and base is $\frac{\pi}{3}$.

Concept Applied

- Area of triangle $=\frac{1}{2}\times\text{Base}\times\text{Height}$

51. Let V and S be the volume and the surface area of a closed cuboid of length = x units, breadth = x units and height = y units respectively.

Then, $V = x^2y \Rightarrow y=\frac{V}{x^2}$ ...(i)

and $S = 2(x^2 + xy + xy) = 2x^2 + 4xy$ ...(ii)

$\Rightarrow S=2x^2+4x\left(\frac{V}{x^2}\right)$ [From (i)]

$\Rightarrow\ S=2x^2+\frac{4V}{x}\ \Rightarrow\ \frac{dS}{dx}=4x-\frac{4V}{x^2}$ ...(iii)

For maximum or minimum of S, $\frac{dS}{dx}=0$

$\therefore\ \frac{dS}{dx}=0\ \Rightarrow\ 4x-\frac{4V}{x^2}=0\Rightarrow V=x^3$

$\Rightarrow\ x^2y=x^3$ $[\because V=x^2y]$

$\Rightarrow\ x=y$

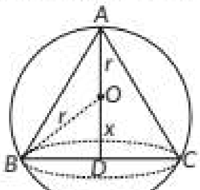
Differentiating (iii) with respect to x, we get

$\frac{d^2S}{dx^2}=4+\frac{8V}{x^3}=4+\frac{8x^2y}{x^3}=4+\frac{8y}{x}$

$\Rightarrow\ \left(\frac{d^2S}{dx^2}\right)_{y=x}=12>0.$

Thus, S is minimum when $x=y$.

52. Let ABC be a cone of maximum volume inscribed in the sphere.



Let OD = x

$\therefore\ BD=\sqrt{r^2-x^2}$

and AD = AO + OD

$=r+x=$ altitude of cone.

Let V be the volume of cone.

$$V=\frac{1}{3}\pi(BD)^2(AD)=\frac{1}{3}\pi(r^2-x^2)(r+x)$$

$\Rightarrow\ \frac{dV}{dx}=\frac{1}{3}\pi\left[(r^2-x^2)+(r+x)(-2x)\right]=\frac{\pi}{3}\left[r^2-3x^2-2rx\right]$

and $\frac{d^2V}{dx^2}=\frac{\pi}{3}[-6x-2r]$

For maximum or minimum value $\frac{dV}{dx}=0$

$\Rightarrow\ r^2-3x^2-2rx=0$

$\Rightarrow\ r^2-3rx+rx-3x^2=0$

$\Rightarrow\ (r-3x)(r+x)=0$

$\Rightarrow\ r=3x$ $[\because r+x\neq 0]$

$\Rightarrow\ x=\frac{r}{3}$

Also, $\left(\frac{d^2V}{dx^2}\right)_{x=\frac{r}{3}}=\frac{\pi}{3}\left[-6\left(\frac{r}{3}\right)-2r\right]$

$=\frac{\pi}{3}[-2r-2r]=\frac{-4}{3}r\pi<0$

$\Rightarrow\ V$ is maximum when $x=\frac{r}{3}$

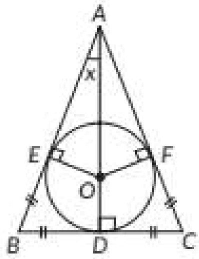
and altitude of cone $=AD=r+x=r+\frac{r}{3}=\frac{4r}{3}$

Also, maximum volume of cone when $x=\frac{r}{3}$

$=\frac{1}{3}\pi\left(r^2-\frac{r^2}{9}\right)\left(r+\frac{r}{3}\right)=\frac{\pi}{3}\left(\frac{8}{9}r^2\right)\left(\frac{4}{3}r\right)$

$=\frac{8}{27}\left(\frac{4}{3}\pi r^3\right)=\frac{8}{27}$ (Volume of sphere) cube units.

53. Let ΔABC be the given triangle and AD is the altitude of the isosceles triangle ABC.



Since, 'r' be the radius of the inscribed circle.

So, OD = OE = OF = r, where O is the centre of the inscribed circle.

AB and AC are the equal sides.

BD = DC ...(i)

BD = BE and CD = CF ...(ii)

From (i) and (ii), BD = BE = DC = CF ...(iii)

Similarly, AE = AF ...(iv)

Perimeter of the triangle ABC = AB + BC + AC

= AE + BE + BD + DC + CF + AF

= 2AE + 4 BD (Using (iii) and (iv))

In right triangle OEA, $AE=\frac{OE}{\tan x}=\frac{r}{\tan x}$ and $AO=\frac{r}{\sin x}$

In right triangle ABD, $BD=AD\tan x$

$=(AO+OD)\tan x=\left(\frac{r}{\sin x}+r\right)\tan x$

Let P be the perimeter of a triangle ABC.

So perimeter, (P) = 2AE + 4BD

$=\frac{2r}{\tan x}+4\left(\frac{r}{\sin x}+r\right)\tan x$

$\Rightarrow\ P(x)=r(2\cot x+4\sec x+4\tan x)$

For maximum or minimum perimeter, $\frac{dP(x)}{dx}=0$

$\Rightarrow\ \frac{dP(x)}{dx}=r(-2\operatorname{cosec}^2x+4\sec x\tan x+4\sec^2x)=0$

$\Rightarrow\ r\left(-\frac{2}{\sin^2x}+\frac{4\sin x}{\cos^2x}+\frac{4}{\cos^2x}\right)=0$

$\Rightarrow\ r\left(\frac{-2\cos^2x+4\sin^3x+4\sin^2x}{\sin^2x\cos^2x}\right)=0$

$\Rightarrow\ -2(1-\sin^2x)+4\sin^3x+4\sin^2x=0$

$\Rightarrow\ 2\sin^3x+3\sin^2x-1=0$

$\Rightarrow\ (\sin x+1)(2\sin^2x+\sin x-1)=0$

sinx cannot be −1 because 'x' cannot be more than 90°.

So, $2\sin^2x+\sin x-1=0$

$\Rightarrow\ (2\sin x-1)(\sin x+1)=0$

Again, sinx cannot be −1.

So $2\sin x-1=0$

$\Rightarrow\ \sin x=\frac{1}{2}\Rightarrow x=30°$

$\frac{d^2P(x)}{dx^2}=r[4\operatorname{cosec}^2x\cot x+4\sec x\tan^2x+4\sec^3x+8\sec^2x\tan x]$

$\therefore\ \left[\frac{d^2P}{dx^2}\right]_{x=30°}>0$

So it is a point of minima for P(x)

Hence, least perimeter $=[P(x)]_{x=30°}=r(2\cot 30°+4\sec 30°+4\tan 30°)$

$=r\left(2\sqrt{3}+4\times\frac{2}{\sqrt{3}}+4\times\frac{1}{\sqrt{3}}\right)=r\left(\frac{18}{\sqrt{3}}\right)=6\sqrt{3}r$

Key Points

- If $f'(c) = 0$ and $f''(c) > 0$, then $x = c$ is a point of local minima.

54. Surface area of cuboid = $2(lb + bh + hl)$

$$=2\left(2x^2+\frac{2x^2}{3}+\frac{x^2}{3}\right)=6x^2$$

Let radius of the sphere be r.
Surface area of sphere = $4\pi r^2$
Therefore, $6x^2 + 4\pi r^2 = k$ (constant) ...(i)
Now, sum of volumes of cuboid and sphere is

$$V=\frac{2}{3}x^3+\frac{4}{3}\pi r^3 \quad ...(ii)$$

Putting the value of r from (i) into (ii), we get

$$V=\frac{2}{3}x^3+\frac{4}{3}\pi\left(\frac{k-6x^2}{4\pi}\right)^{3/2} \quad ...(iii)$$

Differentiating (iii) w.r.t. 'x', we get

$$\frac{dV}{dx}=2x^2+\frac{4}{3}\pi\left(\frac{1}{4\pi}\right)^{3/2}\frac{3}{2}(k-6x^2)^{1/2}(-12x) \quad ...(iv)$$

For minimum or maximum value, $\frac{dV}{dx}=0$

$$\Rightarrow \frac{dV}{dx}=2x^2+\frac{4}{3}\pi\left(\frac{1}{4\pi}\right)^{3/2}\cdot\frac{3}{2}(k-6x^2)^{1/2}(-12x)=0$$

$$\Rightarrow 2x^2=\left(\frac{1}{4\pi}\right)^{1/2}(k-6x^2)^{1/2}(6x)$$

$$\Rightarrow 2x^2=\left(\frac{1}{4\pi}\right)^{1/2}(4\pi r^2)^{1/2}(6x) \quad \text{[From (i)]}$$

$\Rightarrow x = 3r$

Differentiating (iv) w.r.t. 'x', we get

$$\frac{d^2V}{dx^2}=4x-\left(\frac{1}{4\pi}\right)^{\frac{1}{2}}\left[(6)[k-6x^2]^{1/2}+(6x)\frac{(-12x)}{2(k-6x^2)^{1/2}}\right]$$

$$\text{Now, }\left[\frac{d^2V}{dx^2}\right]_{x=3r}=\frac{24\pi r^2+324r^2}{4\pi r}>0$$

Thus, V is minimum at $x = 3r$.
Further, minimum value of sum of their volume

$$=\frac{2}{3}x^3+\frac{4}{3}\pi r^3=\frac{2}{3}x^3+\frac{4}{3}\pi\left(\frac{x}{3}\right)^3 \quad \left[\because r=\frac{x}{3}\right]$$

$$=\frac{2}{3}x^3+\frac{4}{3}\pi\frac{x^3}{27}=\frac{2}{3}x^3\left(1+\frac{2\pi}{27}\right)=\frac{2}{3}x^3\left(1+\frac{44}{189}\right)$$

$$=\frac{2}{3}x^3\cdot\frac{233}{189}=\frac{466}{567}x^3$$

55. We have, $f(x) = \sin x - \cos x$
$\Rightarrow f'(x) = \cos x + \sin x$
For maxima or minima, $f'(x) = 0$
$\Rightarrow \cos x + \sin x = 0 \Rightarrow \tan x = -1$

$$\Rightarrow x=\frac{3\pi}{4},\frac{7\pi}{4}$$

$f''(x) = -\sin x + \cos x$

$$\text{At } x=\frac{3\pi}{4}, f''(x)=-\sin\frac{3\pi}{4}+\cos\frac{3\pi}{4}$$

$$=-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}=\frac{-2}{\sqrt{2}}=-\sqrt{2}$$

$$\text{At } x=\frac{7\pi}{4}, f''(x)=-\sin\frac{7\pi}{4}+\cos\frac{7\pi}{4}$$

$$=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}=\frac{2}{\sqrt{2}}=\sqrt{2}$$

Since $(f''(x)) < 0$ when $x=\frac{3\pi}{4}$

$\therefore$ $f(x)$ has local maxima at $x=\frac{3\pi}{4}$

Since $(f''(x)) > 0$ when $x=\frac{7\pi}{4}$

$\therefore$ $f(x)$ has local minima at $x=\frac{7\pi}{4}$

$\therefore$ Local maximum value at $x=\frac{3\pi}{4}$ is

$$f(x)=\sin\frac{3\pi}{4}-\cos\frac{3\pi}{4}=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}=\frac{2}{\sqrt{2}}=\sqrt{2}$$

Local minimum value at $x=\frac{7\pi}{4}$ is

$$f(x)=\sin\frac{7\pi}{4}-\cos\frac{7\pi}{4}=-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}=\frac{-2}{\sqrt{2}}=-\sqrt{2}$$

Concept Applied

- If $f'(c) = 0$ and $f''(c) < 0$, then $x = c$ is a point of local maxima.
- If $f'(c) = 0$ and $f''(c) > 0$, then $x = c$ is a point of local minima.

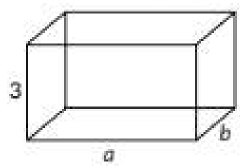
56. Let $P(h, k)$ be the coordinates of the point on given parabola.
$\therefore$ $k = h^2 + 7h + 2$ ...(i)
The distance S of P from the straight line $-3x + y + 3 = 0$ is

$$S=\left|\frac{-3h+k+3}{\sqrt{10}}\right|=\left|\frac{-3h+h^2+7h+2+3}{\sqrt{10}}\right| \quad \text{[From (i)]}$$

$$=\left|\frac{h^2+4h+5}{\sqrt{10}}\right| \quad \therefore \quad S=\frac{f(h)}{\sqrt{10}}$$

$\Rightarrow$ S will be maximum or minimum according as $f(h)$ is maximum or minimum.
Since, $f(h) = h^2 + 4h + 5$
$f'(h) = 2h + 4$
For maxima or minima, $f'(h) = 0 \Rightarrow 2h + 4 = 0 \Rightarrow h = -2$
Also, $f''(h) = 2 > 0$ when $h = -2$
S is minimum at $h = -2$
Putting this value in (i), we get
$k = (-2)^2 + 7(-2) + 2 = 4 - 14 + 2 = -8$
$\therefore$ The required coordinates are $(-2, -8)$

57. Let a m and b m be the sides of the base of the tank.



$\therefore$ Volume of the tank = $a \cdot b \cdot 3 = 75\ m^3$(given)

$$\Rightarrow ab=25\Rightarrow b=\frac{25}{a} \quad ...(i)$$

If C is the total cost in rupees, then

$C = a \times b \times 100 + 2 \times 3 \times a \times 50 + 2 \times 3 \times b \times 50$

$= 100\,ab + 300(a + b) = 100 \times 25 + 300\left(a + \frac{25}{a}\right)$

$\Rightarrow \quad C = 2500 + 300\left(a + \frac{25}{a}\right)$

Differentiating w.r.t. a, we get

$\frac{dC}{da} = 300\left(1 - \frac{25}{a^2}\right)$ and

$\frac{d^2C}{da^2} = 300\left(0 + \frac{25 \times 2}{a^3}\right) = \frac{300 \times 50}{a^3}$

For maximum or minimum cost,

$\frac{dC}{da} = 0 \Rightarrow 1 - \frac{25}{a^2} = 0 \Rightarrow a = 5\text{m}$

and from (i) $b = 5$ m

At $a = 5$; $\frac{d^2C}{da^2} > 0 \Rightarrow C$ is minimum.

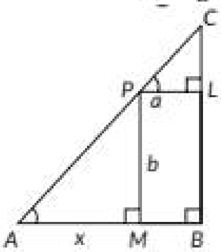
Hence, the least cost of the tank is

$C = \left[2500 + 300\left(5 + \frac{25}{5}\right)\right] = [2500 + 3000] = ₹5500.$

Concept Applied

- Volume of Cuboid $= l \times b \times h$

58. Let P be any point on the hypotenuse of the given right triangle.



Let $PL = a$, $PM = b$ and $AM = x$.

Clearly, ΔCPL and ΔPAM are similar

$\therefore \quad \frac{PL}{CL} = \frac{AM}{PM} \Rightarrow CL = \frac{PL \cdot PM}{AM} = \frac{a \cdot b}{x}$

Now $AB = x + a$ and $BC = b + CL = b + \frac{ab}{x}$.

From right ΔABC, $AC^2 = AB^2 + BC^2$

Taking $l = AC^2$

$\therefore \quad l = (x+a)^2 + \left(b + \frac{ab}{x}\right)^2 = (x+a)^2 + b^2\left(1 + \frac{a}{x}\right)^2$

Differentiating w.r.t. x, we get

$\frac{dl}{dx} = 2(x+a) + b^2 2\left(1 + \frac{a}{x}\right) \cdot \frac{-a}{x^2}$

$= 2(x+a) - \frac{2ab^2(x+a)}{x^3} = 2(x+a)\left[1 - \frac{ab^2}{x^3}\right]$

and $\frac{d^2l}{dx^2} = 2 \cdot 1\left[1 - \frac{ab^2}{x^3}\right] + 2(x+a) \cdot \frac{3ab^2}{x^4}$

$\therefore$ For maximum and minimum value of l,

$\frac{dl}{dx} = 0 \Rightarrow x + a = 0$ or $1 - \frac{ab^2}{x^3} = 0$

As $x = AM \neq 0$ $\therefore$ Reject $x + a = 0$

$\therefore \quad x^3 = ab^2 \Rightarrow x = a^{1/3}b^{2/3}$

For this value of x, clearly $\frac{d^2l}{dx^2} > 0$

$\therefore$ l and consequently the hypotenuse AC is minimum (least).

Hence, the least value of AC is given by

$AC = \sqrt{(x+a)^2 + b^2\left(1 + \frac{a}{x}\right)^2}$ where $x = a^{1/3}b^{2/3}$

$= \sqrt{(x+a)^2 + \frac{b^2}{x^2}(x+a)^2} = (x+a)\sqrt{1 + \frac{b^2}{x^2}}$

$= \left(\frac{x+a}{x}\right)\sqrt{b^2 + x^2} = \left(\frac{a^{1/3}b^{2/3} + a}{a^{1/3}b^{2/3}}\right)\sqrt{b^2 + a^{2/3}b^{4/3}}$

$= \frac{a^{1/3}(b^{2/3} + a^{2/3})}{a^{1/3}b^{2/3}} \cdot b^{2/3}\sqrt{b^{2/3} + a^{2/3}} = (a^{2/3} + b^{2/3})^{3/2}$

59. Let r and h be the radius and height of the cylindrical can respectively.

Therefore, the total surface area of the closed cylinder is given by

$S = 2\pi rh + 2\pi r^2 = 2\pi r(r + h)$ ...(i)

Given volume of the can $= 128\pi\ \text{cm}^3$

Also volume $(V) = \pi r^2 h$ ...(ii)

$\Rightarrow \quad \pi r^2 h = 128\pi \Rightarrow h = \frac{128}{r^2}$ ...(iii)

Putting the value of h in equation (i), we get

$S = 2\pi r\left(r + \frac{128}{r^2}\right) = 2\pi r^2 + \frac{256}{r}\pi$ ...(iv)

Differentiating (iv) w.r.t. r, we get

$\frac{dS}{dr} = 4\pi r - \frac{256\pi}{r^2}$ ...(v)

Substituting $\frac{dS}{dr} = 0$ for critical points, we get

$4\pi r - \frac{256\pi}{r^2} = 0 \Rightarrow r^3 = 64 \Rightarrow r = 4$ cm

Differentiating (v) w.r.t. r, we get

$\frac{d^2S}{dr^2} = 4\pi - 256\pi(-2r^{-3}) = 4\pi + \frac{512}{r^3}\pi \qquad \therefore \left[\frac{d^2S}{dr^2}\right]_{r=4} > 0$

Thus the total surface area of the cylinder is minimum when $r = 4$.

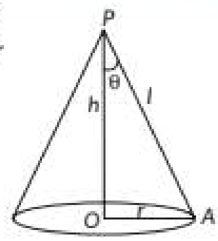
From (iii), we have $h = \frac{128}{r^2} = \frac{128}{16} = 8$

Thus radius = 4 cm and height = 8 cm.

Commonly Made Mistake

- Remember the difference between first derivative test and second derivative test for finding local maxima and local minima.

60. Let θ be the semi-vertical angle of the cone, V its volume, h its height, r base radius and slant height l.



Then from ΔOAP,

$r = l\sin\theta$, $h = l\cos\theta$

Now, $V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi l^2 \sin^2\theta \cdot l\cos\theta$

$= \frac{1}{3}\pi l^3 \sin^2\theta \cdot \cos\theta$

$\Rightarrow \quad \frac{dV}{d\theta}=\frac{1}{3}\pi l^3\,(2\sin\theta\cdot\cos\theta\cdot\cos\theta-\sin^2\theta\cdot\sin\theta)$

$=\frac{1}{3}\pi l^3\sin\theta(2\cos^2\theta-\sin^2\theta)$

and $\frac{d^2V}{d\theta^2}=\frac{1}{3}\pi l^3\,[\cos\theta(2\cos^2\theta-\sin^2\theta)$

$+\sin\theta\,(-4\cos\theta\sin\theta-2\sin\theta\cos\theta)]$

$=\frac{1}{3}\pi l^3[\cos\theta(2\cos^2\theta-\sin^2\theta)-6\sin^2\theta\cos\theta]$

For maximum or minimum value of V,

$\frac{dV}{d\theta}=0\Rightarrow\sin\theta(2\cos^2\theta-\sin^2\theta)=0$

$\Rightarrow \quad \sin\theta=0$ or $2\cos^2\theta-\sin^2\theta=0$

$\Rightarrow \quad 2\cos^2\theta-(1-\cos^2\theta)=0$ [Note : $\sin\theta\neq0$ as $\theta\neq0$]

$\Rightarrow \quad \cos^2\theta=\frac{1}{3}\Rightarrow\cos\theta=\frac{1}{\sqrt{3}} \Rightarrow \theta=\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$

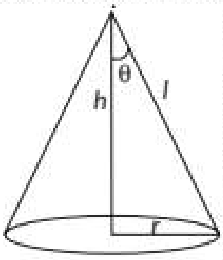
For $\cos\theta=\frac{1}{\sqrt{3}} \Rightarrow \sin\theta=\frac{\sqrt{2}}{\sqrt{3}}$

$\therefore \quad \frac{d^2V}{d\theta^2}=\frac{1}{3}\pi l^3\left[\frac{1}{\sqrt{3}}\left(2\cdot\frac{1}{3}-\frac{2}{3}\right)-6\cdot\frac{2}{3}\cdot\frac{1}{\sqrt{3}}\right]$

$=\frac{1}{3}\pi l^3\left(-\frac{4}{\sqrt{3}}\right)<0$

$\therefore \quad V$ is maximum for $\theta=\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

61. Let r, h, l, V and S be respectively the base radius, height, slant height, volume and curved surface area of the cone. Then,



$l^2=r^2+h^2$,

$V=\frac{1}{3}\pi r^2h$ ... (i)

and $S=\pi rl=\pi r\sqrt{r^2+h^2}$

$\Rightarrow \quad S^2=\pi^2r^2(r^2+h^2)$

$=\pi^2\frac{3V}{\pi h}\left(\frac{3V}{\pi h}+h^2\right)$ [Using (i)]

$=3\pi V\left(\frac{3V}{\pi h^2}+h\right)$

For S to be least, S^2 is also least.

$\therefore \quad \frac{dS^2}{dh}=3\pi V.\left(\frac{-6V}{\pi h^3}+1\right)$ and

$\frac{d^2S^2}{dh^2}=3\pi V\left(\frac{-6V}{\pi}\right)\cdot\frac{-3}{h^4}=\frac{54V^2}{h^4}$

For maximum or minimum S (and so S^2),

$\frac{dS^2}{dh}=0\Rightarrow 6V=\pi h^3$

$\Rightarrow \quad h=\left(\frac{6V}{\pi}\right)^{1/3}$ ... (ii)

For this value of h, $\frac{d^2S^2}{dh^2}=\frac{54V^2}{h^4}>0$

$\Rightarrow \quad S^2$ and therefore S is least.

$\cot\theta=\frac{h}{r}=\frac{h}{\sqrt{3V/\pi h}}=\sqrt{\frac{\pi}{3V}}\cdot h^{3/2}$ [From (i)]

$\Rightarrow \quad \cot\theta=\sqrt{\frac{\pi}{3V}}\sqrt{\frac{6V}{\pi}}=\sqrt{2}$ [From (ii)]

$\Rightarrow \quad$ The semi vertical angle, $\theta=\cot^{-1}\sqrt{2}$

Answer Tips

- $\cot\theta=\frac{\text{Base}}{\text{Perpendicular}}$

62. Let a be the side of the given square and r be the radius of the circle.

By hypothesis

$4a+2\pi r=k$

$\Rightarrow \quad a=\frac{k-2\pi r}{4}$ ...(i)

Let A = Sum of areas of the circle and the square

$\Rightarrow \quad A=\pi r^2+a^2=\pi r^2+\frac{1}{16}(k-2\pi r)^2$ [Using (i)]

$\Rightarrow \quad \frac{dA}{dr}=2\pi r+\frac{1}{16}\cdot2(k-2\pi r)\cdot(-2\pi)$

$=2\pi r-\frac{\pi}{4}(k-2\pi r)$

and $\frac{d^2A}{dr^2}=2\pi-\frac{\pi}{4}(0-2\pi)=2\pi+\frac{\pi^2}{2}$

For maxima or minima,

$\frac{dA}{dr}=0\Rightarrow 2\pi r-\frac{\pi}{4}(k-2\pi r)=0$

$\Rightarrow \quad 8r-k+2\pi r=0\Rightarrow(8+2\pi)r=k \Rightarrow r=\frac{k}{2\pi+8}$

For this value of r, $\frac{d^2A}{dr^2}=2\pi+\frac{\pi^2}{2}>0$

$\therefore \quad A$ is minimum (least), when $r=\frac{k}{2\pi+8}$

From (i), $a=\frac{k-2\pi\cdot\frac{k}{2\pi+8}}{4}$

$=\frac{k}{4}\cdot\left(\frac{2\pi+8-2\pi}{2\pi+8}\right)=\frac{2k}{2\pi+8}=2r$

$\therefore \quad$ Area is least, when $a=2r$.

63. Let r and h be the base radius and height of the cylinder respectively and volume of cylinder, $V=\pi r^2h$

$\Rightarrow \quad h=\frac{V}{\pi r^2}$ ... (i)

Total surface area of the cylinder, $S=2\pi rh+\pi r^2$

$\Rightarrow \quad S=2\pi r\left(\frac{V}{\pi r^2}\right)+\pi r^2$ [By using (i)]

$\Rightarrow \quad S=\frac{2V}{r}+\pi r^2$

On differentiating w.r.t. r both sides, $\frac{dS}{dr}=-\frac{2V}{r^2}+2\pi r$

Again differentiating w.r.t. r both sides, $\frac{d^2S}{dr^2}=\frac{4V}{r^3}+2\pi$

For maxima or minima, $\frac{dS}{dr}=0$

$\Rightarrow \quad -\frac{2V}{r^2}+2\pi r=0 \;\Rightarrow\; 2\pi r=\frac{2V}{r^2}$

$\Rightarrow \quad \pi r^3=V \;\Rightarrow r=\left(\frac{V}{\pi}\right)^{1/3}$

$\therefore \quad \left[\frac{d^2V}{dh^2}\right]_{r=\left(\frac{V}{\pi}\right)^{1/3}}=4V\left(\frac{\pi}{V}\right)+2\pi=6\pi>0$

So, S is minimum at $r=\left(\frac{V}{\pi}\right)^{1/3}$

Now, $\pi r^3 = V \Rightarrow \pi r^3 = \pi r^2 h \Rightarrow r = h$

Hence, the cylinder of a given volume which is open at the top has minimum total surface area, when its height is equal to the radius of its base.

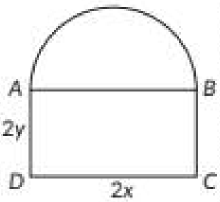
64. Let $ABCD$ be a rectangle and let the semi-circle is described on the side AB as its diameter.

Let $AB = 2x$ and $AD = 2y$. Let $P = 10$ m be the given perimeter of window.

Therefore, $10 = 2x + 4y + \pi x$

$\Rightarrow \quad 4y = 10 - 2x - \pi x \quad$ ...(i)

Area of the window,



$A=(2x)(2y)+\frac{1}{2}\pi x^2$

$\Rightarrow \quad A=4xy+\frac{1}{2}\pi x^2$

$\Rightarrow \quad A=10x-2x^2-\pi x^2+\frac{1}{2}\pi x^2$ [using (i)]

$\Rightarrow \quad A=10x-2x^2-\frac{1}{2}\pi x^2$

On differentiating w.r.t. x, $\frac{dA}{dx}=10-4x-\pi x$

Again differentiating w.r.t. x, $\frac{d^2A}{dx^2}=-(4+\pi)$

For maxima or minima,

$\frac{dA}{dx}=0 \Rightarrow 10-4x-\pi x=0 \Rightarrow x=\frac{10}{4+\pi}$

$\left[\frac{d^2A}{dx^2}\right]_{x=\frac{10}{4+\pi}}=-(4+\pi)<0$

So, A is maximum at $x=\left(\frac{10}{4+\pi}\right)$m.

Now, length of the window is $2x=\left(\frac{20}{4+\pi}\right)$m and width is $2y=\left(\frac{10}{4+\pi}\right)$m.

Concept Applied

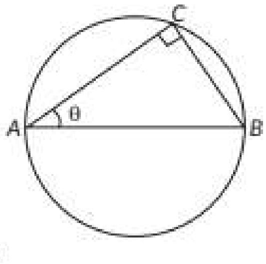
- Area of rectangle = length × breadth

  Area of semicircle $=\frac{1}{2}\pi r^2$

65. Here BA is a diameter of the given circle, of radius = r.

Let $\angle CAB = \theta$

Also $\angle ACB=\frac{\pi}{2}$



Now $AC = AB \cos\theta = 2r \cos\theta$

$BC = AB \sin\theta = 2r\sin\theta$

Let Area of $\Delta ABC=\frac{1}{2}\cdot AC\cdot BC$

$=\frac{1}{2}\cdot 2r\cos\theta\cdot 2r\sin\theta=r^2\cdot\sin 2\theta$

$\Rightarrow \quad \frac{d\Delta}{d\theta}=r^2\cdot 2\cos 2\theta$ and $\frac{d^2\Delta}{d\theta^2}=-r^2\cdot 4\sin 2\theta$

For maxima or minima,

$\frac{d\Delta}{d\theta}=0 \Rightarrow \cos 2\theta=0$

$\Rightarrow \quad 2\theta=\frac{\pi}{2} \Rightarrow \theta=\frac{\pi}{4}$

and $\left[\frac{d^2\Delta}{d\theta^2}\right]_{\theta=\frac{\pi}{4}}=-4r^2\sin\frac{\pi}{2}=-4r^2<0$

Hence, area of ΔABC is maximum when

$\angle CAB=\theta=\frac{\pi}{4}=\angle ABC \quad \left[\because \quad \angle ACB=\frac{\pi}{2}\right]$

$\Rightarrow \quad \Delta ABC$ is isosceles.

Answer Tips

- Chain rule of derivative : $\frac{dt}{dx}=\frac{dt}{du}\cdot\frac{du}{dx}$

66. The given parabola is

$y^2 = 4ax$ ...(i)

Let $Q(11a, 0)$.

Any point on (i) is $P(at^2, 2at)$

$\therefore \quad PQ^2 = (at^2 - 11a)^2 + (2at - 0)^2$

Let $l = PQ^2 = a^2t^4 - 18a^2t^2 + 121a^2$

$\Rightarrow \quad \frac{dl}{dt}=4a^2t^3-36a^2t$ and $\frac{d^2l}{dt^2}=12a^2t^2-36a^2$

For maximum or minimum value of l,

$\frac{dl}{dt}=0 \Rightarrow 4a^2t(t^2-9)=0 \Rightarrow t=0, 3, -3$

For $t=0, \frac{d^2l}{dt^2}=-36a^2<0$

This corresponds to a maximum value of l.

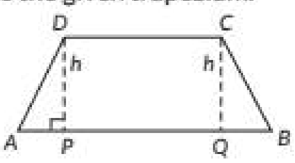
Both for t = 3 and − 3,

$\frac{d^2l}{dt^2}=12a^2\cdot 9-36a^2=72a^2>0$

$\therefore \quad$ This corresponds to a minimum value of l i.e., of PQ^2 and therefore of PQ.

Thus, there are two such points P with coordinates, $(9a, 6a)$ and $(9a, -6a)$ nearest to the given point Q.

67. Let $ABCD$ be the given trapezium.



Then $AD = DC = CB = 10$ cm
In ΔAPD and ΔBQC
$DP = CQ = h$
$AD = BC = 10$ cm
$\angle DPA = \angle CQB = 90°$
$\therefore \quad \Delta APD \cong \Delta BQC$ (by R.H.S. congruency)
$\Rightarrow \quad AP = QB = x$ cm (Say)
$\therefore \quad AB = AP + PQ + QB$
$= x + 10 + x = (2x + 10)$cm
Also from ΔAPD, $AP^2 + PD^2 = AD^2$
$\Rightarrow \quad x^2 + h^2 = 10^2$
$\Rightarrow \quad h = \sqrt{100 - x^2}$ ...(i)
Now, area A of this trapezium is given by

$$A = \frac{1}{2}(AB + DC)\cdot h = \frac{1}{2}(2x + 10 + 10)h$$

$$= (x + 10)\cdot\sqrt{100 - x^2} \quad \text{...(ii)} \quad [\text{Using (i)}]$$

Differentiating w.r.t. x, we get

$$\frac{dA}{dx} = 1\cdot\sqrt{100 - x^2} + (x + 10)\cdot\frac{1}{2\sqrt{100 - x^2}}\cdot(-2x)$$

$$= \frac{100 - x^2 - x^2 - 10x}{\sqrt{100 - x^2}} = \frac{-2(x^2 + 5x - 50)}{\sqrt{100 - x^2}}$$

$$= \frac{-2(x + 10)(x - 5)}{\sqrt{100 - x^2}}$$

For maximum or minimum value of A, $\frac{dA}{dx} = 0$
$\Rightarrow \quad (x + 10)(x - 5) = 0$
$\Rightarrow \quad x = 5$ (Reject $x = -10$ as $x \neq 0$)

For this value of x, $\frac{dA}{dx}$ changes sign from positive to negative.
$\therefore \quad A$ is maximum at $x = 5$.
From (ii), the maximum value of $A = (5 + 10)\cdot\sqrt{100 - 5^2}$

$$= 15\sqrt{75} = 75\sqrt{3} \text{ sq.cm.}$$

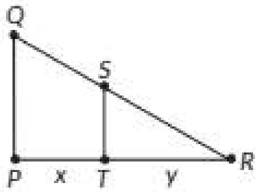
Key Points

- Area of Trapezium $= \frac{1}{2}(a + b)h$, where a and b are parallel side and h is the height of the trapezium.

CBSE Sample Questions

1. Let PQ represent the height of the street light from the ground. At any time t seconds, let the man represent as ST of height 1.6 m be at a distance of x m from PQ and the length of his shadow TR be y m.

Using similarity of triangles, we have $\frac{4}{1.6} = \frac{x + y}{y}$ **(1/2)**
$\Rightarrow \quad 3y = 2x$



Differentiating both sides w.r.t. t, we get $3\frac{dy}{dt} = 2\frac{dx}{dt}$

$\Rightarrow \quad \frac{dy}{dt} = \frac{2}{3}\times 0.3 \Rightarrow \frac{dy}{dt} = 0.2$ **(1/2)**

At any time t seconds, the tip of his shadow is at a distance of $(x + y)$ m from PQ.
The rate at which the tip of his shadow moving

$= \left(\frac{dx}{dt} + \frac{dy}{dt}\right)$m/s $= 0.5$ m/s **(1/2)**

The rate at which his shadow is lengthening

$= \frac{dy}{dt}$ m/s $= 0.2$ m/s **(1/2)**

2. **(b):** We have, $f(x) = x^2 - 4x + 6$
$\Rightarrow \quad f'(x) = 2x - 4$
$\because \quad f(x)$ is strictly increasing.
$\therefore \quad f'(x) > 0$
$\Rightarrow \quad 2x - 4 > 0 \Rightarrow x > 2$
$\Rightarrow \quad x \in (2, \infty)$ **(1)**

3. **(b):** We have, $f(x) = 2x^3 - 3x^2 - 36x + 7$
$\Rightarrow \quad f'(x) = 6x^2 - 6x - 36 = 6(x^2 - x - 6)$
$= 6(x - 3)(x + 2)$
$f'(x) = 0 \Rightarrow (x - 3)(x + 2) = 0, x = -2, 3$

$\therefore \quad f(x)$ is strictly increasing in $(-\infty, -2) \cup (3, \infty)$ and strictly decreasing in $(-2, 3)$. **(1)**

4. **(b):** We have, $f(x) = x + \cos x + b$
$\Rightarrow \quad f'(x) = 1 - \sin x \Rightarrow f'(x) \geq 0 \ \forall \ x \in R$
$\Rightarrow \quad$ No such value of b exists **(1)**

5. We have, $f(x) = \tan x - 4x$
$\Rightarrow \quad f'(x) = \sec^2 x - 4$ **(1)**

(a) For $f(x)$ to be strictly increasing, $f'(x) > 0$
$\Rightarrow \quad \sec^2 x - 4 = 0 \Rightarrow \sec^2 x > 4$

$$\Rightarrow \cos^2 x < \frac{1}{4} \Rightarrow \cos^2 x < \left(\frac{1}{2}\right)^2$$

$$\Rightarrow -\frac{1}{2} < \cos x < \frac{1}{2} \Rightarrow \frac{\pi}{3} < x < \frac{\pi}{2} \quad \left(\because x \in \left(0, \frac{\pi}{2}\right)\right)$$ **(1)**

(b) For $f(x)$ to be strictly decreasing, $f'(x) < 0$
$\Rightarrow \quad \sec^2 x - 4 < 0 \Rightarrow \sec^2 x < 4$

$$\Rightarrow \cos^2 x > \frac{1}{4} \Rightarrow \cos^2 x > \left(\frac{1}{2}\right)^2$$

$$\Rightarrow \cos x > \frac{1}{2}\left[\because x \in \left(0, \frac{\pi}{2}\right)\right] \Rightarrow 0 < x < \frac{\pi}{3}$$ **(1)**

6. **(c):** We have, $f(x) = 2\cos x + x$
$\Rightarrow \quad f'(x) = -2\sin x + 1$
$\Rightarrow \quad f''(x) = -2\cos x$
For critical points, $f'(x) = 0$
$\Rightarrow \quad -2\sin x + 1 = 0$

$$\Rightarrow \sin x = \frac{1}{2} = \sin\left(\frac{\pi}{6}\right) \quad \left(\because x \in \left[0, \frac{\pi}{2}\right]\right)$$

$\Rightarrow x=\frac{\pi}{6}$

$f''(x)\left(\text{at } x=\frac{\pi}{6}\right)=-2\cos\frac{\pi}{6}=-\sqrt{3}<0$

So, $x=\frac{\pi}{6}$ is the point of maxima.

Now, $f(0)=2$ and $f\left(\frac{\pi}{2}\right)=\frac{\pi}{2}=1.57$

$\Rightarrow$ Least value of $f(x)=1.57$ *i.e.*, $\frac{\pi}{2}$ **(1)**

7. (c) : We have, $f(x)=(10+x)\sqrt{100-x^2}$

Which will give some real area if $-10<x<10$

$\Rightarrow f'(x)=\frac{(10+x)\times(-2x)}{2\sqrt{100-x^2}}+\sqrt{100-x^2}\times 1$

$\Rightarrow f'(x)=\frac{-2x^2-10x+100}{\sqrt{100-x^2}}$

For critical points, put $f'(x)=0$

$\Rightarrow x^2+5x-50=0$

$\Rightarrow (x+10)(x-5)=0$

$\Rightarrow x=-10$ or $5 \Rightarrow x=5$ $[\because -10<x<10]$

Now, $f''(x)$

$$=\frac{(\sqrt{100-x^2})(-4x-10)+(2x^2+10x-100)\times\frac{1}{2}\frac{(-2x)}{\sqrt{100-x^2}}}{(100-x^2)}$$

$$=\frac{2x^3-300x-1000}{(100-x^2)^{3/2}}\Rightarrow f''(5)=\frac{-30}{\sqrt{75}}<0$$

$\therefore$ Maximum area of trapezium

$=(10+5)(\sqrt{75})=75\sqrt{3}\text{ cm}^2$ **(1)**

8. (c) : Let $f(x)=[x(x-1)+1]^{1/3}, 0\le x\le 1$

$\Rightarrow f'(x)=\frac{2x-1}{3(x^2-x+1)^{2/3}}$

For critical points, put $f'(x)=0$

$\Rightarrow x=\frac{1}{2}\in[0,1]$

Now, $f(0)=1, f\left(\frac{1}{2}\right)=\left(\frac{3}{4}\right)^{1/3}$ and $f(1)=1$

$\therefore$ Maximum value of $f(x)$ is 1. **(1)**

9. (d) : Let F be the fuel cost per hour and v be the speed of train in km/hr.

According to question, we have,

$F\propto v^2\Rightarrow F=kv^2$, where k is proportionality constant

$\Rightarrow 48=k(16)^2\Rightarrow k=\frac{3}{16}$ **(1)**

10. (b) : Let total cost of running the train be C.

Then, $C=\frac{3}{16}v^2t+1200t$

Now, distance covered = 500 km $\Rightarrow$ Time $=\frac{500}{v}$ hrs

$\therefore$ Total cost of running the train for 500 km

$=\frac{3}{16}v^2\left(\frac{500}{v}\right)+1200\left(\frac{500}{v}\right)$

$\Rightarrow C=\frac{375}{4}v+\frac{600000}{v}$ **(1)**

11. (c) : We have, $\frac{dC}{dv}=\frac{375}{4}-\frac{600000}{v^2}$

Put $\frac{dC}{dv}=0\Rightarrow v^2=\frac{600000\times 4}{375}=6400$

$\Rightarrow v=80$ km/h

$\frac{d^2C}{dv^2}=\frac{2\times 600000}{v^3}>0$, for $v=80$

$\therefore$ Most economical speed is 80 km/h. **(1)**

12. (c) : Fuel cost for running the train for 500 km

$=\frac{3}{16}v^2\left(\frac{500}{v}\right)$

$=\frac{375}{4}v=\frac{375}{4}\times 80=₹7500$ **(1)**

13. (d) : Total cost for running the train for 500 km

$=\frac{375}{4}v+\frac{600000}{v}$

$=\frac{375\times 80}{4}+\frac{600000}{80}=₹15000$ **(1)**

14. (i) (b) : We have, perimeter of floor = 200 m

$\Rightarrow 2x+2\pi\left(\frac{y}{2}\right)=200$

$\Rightarrow 2x+\pi y=200$ ... (i) **(1)**

(ii) (a) : Area of rectangular region $(A)=xy$

$=x\left(\frac{200-2x}{\pi}\right)$ [Using (i)]

$=\frac{2}{\pi}(100x-x^2)$ **(1)**

(iii) (c) : We have, $A=\frac{2}{\pi}(100x-x^2)$

$\Rightarrow \frac{dA}{dx}=\frac{2}{\pi}(100-2x)$

For maximum or minimum, $\frac{dA}{dx}=0$

$\Rightarrow 100-2x=0 \Rightarrow x=50$

Now, $\left[\frac{d^2A}{dx^2}\right]_{x=50}=-\frac{4}{\pi}<0$

Thus, A is maximum at $x=50$.

Thus, maximum value of $A=\frac{2}{\pi}(5000-2500)$

$=\frac{5000}{\pi}\text{m}^2$ **(1)**

(iv) (a) : Let P be the area of the whole floor.

Then, $P=xy+\pi\left(\frac{y}{2}\right)^2=xy+\frac{\pi}{4}y^2=y\left(x+\frac{\pi}{4}y\right)$

$=\left(\frac{200-2x}{\pi}\right)\left(\frac{200+2x}{4}\right)$ [Using (i)]

$=\frac{40000-4x^2}{4\pi}=\frac{10000-x^2}{\pi}$

$\therefore \frac{dP}{dx}=-\frac{2x}{\pi}$

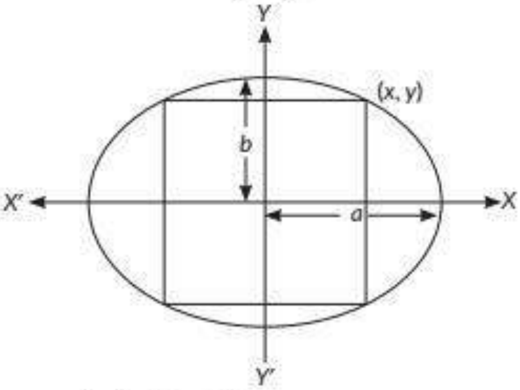
For maximum or minimum, $\frac{dP}{dx}=0 \Rightarrow x=0$

Now, $\frac{d^2P}{dx^2}=\frac{-2}{\pi}<0$

So, P is maximum at $x=0$ m. **(1)**

(v) (d) **(1)**

15. (i) Given ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



(1/2)

Let $(x,y)=\left(x,\frac{b}{a}\sqrt{a^2-x^2}\right)$ be the upper right vertex of the rectangle.

The area function $A=2x\times2\frac{b}{a}\sqrt{a^2-x^2}$

$=\frac{4b}{a}x\sqrt{a^2-x^2}, x\in(0,a)$ **(1/2)**

(ii) The first derivative of function is

$\frac{dA}{dx}=\frac{4b}{a}\left[x\times\frac{-x}{\sqrt{a^2-x^2}}+\sqrt{a^2-x^2}\right]$

$=\frac{4b}{a}\times\frac{a^2-2x^2}{\sqrt{a^2-x^2}}=-\frac{4b}{a}\times\frac{2\left(x+\frac{a}{\sqrt{2}}\right)\left(x-\frac{a}{\sqrt{2}}\right)}{\sqrt{a^2-x^2}}$ **(1/2)**

To find the critical point, put $\frac{dA}{dx}=0$

$\Rightarrow x=\frac{a}{\sqrt{2}}$

So, $x=\frac{a}{\sqrt{2}}$ is the critical point. **(1/2)**

(iii) For the values of x less than $\frac{a}{\sqrt{2}}$ and close to $\frac{a}{\sqrt{2}}, \frac{dA}{dx}>0$ and for the values of x greater than $\frac{a}{\sqrt{2}}$ and close to $\frac{a}{\sqrt{2}}, \frac{dA}{dx}<0$. **(1)**

Hence, by the first derivative test, there is a local maximum at the critical point $x=\frac{a}{\sqrt{2}}$.

Since there is only one critical point, therefore, the area of the soccer field is maximum at this critical point $x=\frac{a}{\sqrt{2}}$. **(1)**

Thus, for maximum area of the soccer field, its length should be $a\sqrt{2}$ and its width should be $b\sqrt{2}$.

OR

$A=2x\times2\frac{b}{a}\sqrt{a^2-x^2}, x\in(0,a)$

Squaring both sides, we get

$Z=A^2=\frac{16b^2}{a^2}x^2(a^2-x^2)=\frac{16b^2}{a^2}(x^2a^2-x^4), x\in(0,a)$ **(1/2)**

A is maximum when Z is maximum

To find the critical point, put $\frac{dz}{dx}=0$ **(1/2)**

$\frac{dZ}{dx}=\frac{16b^2}{a^2}(2xa^2-4x^3)=\frac{32b^2}{a^2}x(a+\sqrt{2}x)(a-\sqrt{2}x)$

To find the critical point, put $\frac{dZ}{dx}=0 \Rightarrow x=\frac{a}{\sqrt{2}}$

The second derivative is ; $\frac{d^2Z}{dx^2}=\frac{32b^2}{a^2}(a^2-6x^2)$

$\therefore \left(\frac{d^2Z}{dx^2}\right)_{x=\frac{a}{\sqrt{2}}}=\frac{32b^2}{a^2}(a^2-3a^2)=-64b^2<0$ **(1/2)**

Hence, by the second derivative test, there is a local maximum value of Z at the critical point $x=\frac{a}{\sqrt{2}}$. **(1/2)**

Since there is only one critical point, therefore, Z is maximum at $x=\frac{a}{\sqrt{2}}$, hence, A is maximum at $x=\frac{a}{\sqrt{2}}$.

Thus, for maximum area of the soccer field, its length should be $a\sqrt{2}$ and its width should be $b\sqrt{2}$.